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CONTENTS

NEW PROOFS OF SOME THEOREMS OF HARDY
BY BANACH SPACE METHODS

Angus E. Taylor

FOREWORD

In 1914 the famous English mathematician, Godfrey Harold Hardy
(born 1877, died 1947), published a paper containing the theorems which
form the subject of the present paper. To understand the theorems it is
necessary to know something of the theory of functions of a complex
variable. Also, in order to appreciate the interest which attaches to
these theorems, it is desirable to know the theorems of classical
function-theory of which Hardy's theorems are generalizations. The first
of these classical theorems is the following:

*Let $f(z)$ be an analytic function which is without singularities in
the circle $|z| < R$. Let $M(r)$ denote the maximum value of $|f(z)|$ on the
circle $|z| = r$, where $0 \leqq r < R$. Then $M(r)$ does not decrease as r increases.*

Before stating the second classical theorem we shall define the term
convex function. If $\varphi(x)$ is a real function of the real variable x on
the interval $a < x < b$, consider the graph of the equation $y = \varphi(x)$.
Suppose the function is such that, whenever we choose two points of the
graph and join them by a straight line segment, the portion of the
graph between the points never rises above the line segment. Then we
say that the function $\varphi(x)$ is convex.

The second classical theorem is due to J. Hadamard. It deals with
$f(z)$ and $M(r)$ as given in the first theorem. Now we assume that $f(z)$ is
not identically zero, so that $M(r) > 0$ when $r > 0$. *Then Hadamard's theorem
states that, when $0 < r < R$, $\log M(r)$ is a convex function of $\log r$.*

The convexity of a function can be expressed by inequalities relating
to three points in order along the graph. In terms of inequalities
Hadamard's theorem states that if $0 < r_1 < r_2 < r_3 < R$, then

$$(\log r_3/r_1) \, \log M(r_2) \leqq (\log r_3/r_2) \, \log M(r_1) + (\log r_2/r_1) \, \log M(r_3).$$

Because of the three radii mentioned here, the theorem is usually called
the three circles theorem.

Now for Hardy's generalizations. Let us define

$$\mathfrak{M}_p(r) = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{\frac{1}{p}},$$

where $0 \leqq r < R$ and p is any positive number. *Hardy discovered and proved
that the above theorems are still true if we replace $M(r)$ by $\mathfrak{M}_p(r)$ in*

the statements of the theorems. The link between the classical theorems and those of Hardy is in the fact that $\mathfrak{M}_p(r) \to M(r)$ as $p \to \infty$.

We now offer a new approach to Hardy's theorems. It is a sophisticated approach, for it presumes some considerable familiarity with the modern practice of treating functions as 'vectors' in a Banach space, and applying to function spaces the concepts of geometry and analysis as they were developed originally for Euclidean space and ordinary function-theory. If it were only a question of proving Hardy's theorems as simply as possible, it would be ridiculous to devise such elaborate and sophisticated methods. But, to a person having already some familiarity with the development of the theory of functions in Banach spaces, our method of proving Hardy's theorems will be both easy and illuminating. Our proof is, in substance, the observation, first, that the two classical theorems remain true when the values of $f(z)$ lie in a complex Banach space (absolute values being replaced by norms), and second, that Hardy's theorems are *special instances* of the classical theorems when they are thus generalized.

1. *Introduction.* We shall be concerned with theorems about functions which are analytic within a circle in the complex plane. For convenience, and without any essential loss of generality, we shall assume that the circle has unit radius and center at the origin.

Definition 1.1. Let $\mathfrak{U}$ denote the class of all complex functions $f(z)$ which are singlevalued and analytic when $|z| < 1$.

Definition 1.2. For $f \in \mathfrak{U}$ and $0 \leq r < 1$ we define

$$M(f;r) = \max_{|z|=r} |f(z)| = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

For an index $p > 0$ we define

$$\mathfrak{M}_p(f;r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

The purpose of this paper is to give a new proof of the following theorem. The theorem is true for all $p > 0$, but for our proof we require that $p \geq 1$. The interest of our proof lies in the fact that it employs the theory of analytic functions with values in a complex Banach space.

Theorem 1.1. *The mean* $\mathfrak{M}_p(f;r)$, *for a fixed* $f \in \mathfrak{U}$, *is a nondecreasing function of* r $(0 \leq r < 1)$. *It is a strictly increasing function of* r *if* $f(z)$ *is not a constant. If* $f(z)$ *is not identically zero,* $\log \mathfrak{M}_p(f;r)$ *is a convex function of* $\log r$ *when* $0 < r < 1$.

With the exception of the assertion that $\mathfrak{M}_p(f;r)$ is strictly increasing when f is not a constant, this theorem is originally due to Hardy [1].*

If we replace $\mathfrak{M}_p(f;r)$ by $M(f;r)$ in the statement of Theorem 1.1, we have a true theorem, the first part of which is a corollary of

*Numbers in square brackets refer to the bibliography at the end of the paper.

the maximum modulus theorem, while the second part is the Hadamard three circles theorem.

Now it is well known and readily proved that

$$\lim_{p \to \infty} \mathfrak{M}_p(f;r) = M(f;r),$$

so that it is natural to define

$$\mathfrak{M}_\infty(f;r) = M(f;r).$$

When this notation is adopted we see that Theorem 1.1 appears as a natural companion of (and in a sense an extension of) two well known theorems in the classical theory of functions. We shall in fact show that when $p \geq 1$ the assertions of Theorem 1.1 may be obtained as actual instances of the classical theorems when the latter are extended to apply to functions whose values, instead of being complex numbers, are elements of a complex Banach space.

In brief outline our procedure is the following: We first extend the classical theorems to functions with values in a Banach space. Next we consider a special kind of Banach space $\mathfrak{B}$ whose elements are ordinary analytic functions (members of the class $\mathfrak{A}$). By suitable specialization of $\mathfrak{B}$ we are able to associate with each $f \in \mathfrak{A}$ an analytic function *with values in* $\mathfrak{B}$ such that the maximum of the norm of this function on the circle $|z| = r$ is $\mathfrak{M}_p(f;r)$. The desired conclusions then follow.

Unfortunately, this method as it stands requires the limitation $p \geq 1$. Thus it suffers by comparison with proofs of Hardy's theorem by the method of subharmonic functions (see for example F. Riesz [4]). It is, however, interesting as an illustration of the usefulness of extending theorems of classical analysis to Banach spaces. We observe that our method yields an easy proof that $\mathfrak{M}_p(f;r)$ is strictly increasing when f is not a constant.

Finally, it is worth emphasizing that the study of Banach spaces whose elements are members of the class $\mathfrak{A}$, subject to certain restrictions such as we impose in §3, is an interesting and fruitful pursuit. Some of the author's investigations of such spaces, including the subject matter of the present paper, were reported on in an invited address before the London Mathematical Society, February 19, 1948.

2. *Vectorvalued analytic functions.* Let Z be the complex plane, and let X be a complex Banach space. We are concerned with functions defined on a subset of Z and having values in X.

Definition 2.1. If D is a connected open set in Z we shall say that a function f is of type (A) on D to X if it satisfies the following conditions:

(a) f is defined on D, with values in X; f need not be singlevalued, but we assume that in some neighborhood of each point of D the values

of f can be sorted out into singlevalued branches each of which is analytic in the neighborhood.

(b) $\|f(z)\|$ is singlevalued in D (it then follows from (a) that $\|f(z)\|$ is continuous in D).

Theorem 2.1 (*The maximum modulus theorem*). *Let D be an open connected set in Z, and let f be of type (A) on D to X. Then, except in the case that $\|f(z)\|$ is constant on D, $\|f(z)\|$ cannot attain an absolute maximum in D.*

Proof: Suppose $\|f(z)\|$ attains an absolute maximum at a point z_0 in D and that it is not constant in D. Let $N = \|f(z_0)\|$, and let S be the set of points in D at which $\|f(z)\| < N$. Then S is open and nonempty. The set $D-S$ is also nonempty, for it contains z_0. Now $D-S$ has no limit point in S, since S is open. Hence, since D is connected, S must have a limit point z_1 in $D-S$. Then $\|f(z_1)\| = N$. For some $r > 0$ the circle C: $|z-z_1| = r$, together with its interior lies in D, and on C there is a point z_2 of S. We may assume that r is small enough so that each branch of $f(z)$ is analytic within and on C. Then, for any branch, Cauchy's formula gives us

$$f(z_1) = \frac{1}{2\pi} \int_0^{2\pi} f(z_1 + re^{i\theta})d\theta.$$

Hence

$$N = \|f(z_1)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|f(z_1 + re^{i\theta})\|d\theta \leq N.$$

Thus the average value of the function $\|f(z)\|$ on C is N. Since the function is continuous and never greater than N on C, it must equal N at all points of C, and in particular at z_2. This contradicts the fact that $z_2 \in S$, however. Hence our initial supposition was false and the proof is complete.

Theorem 2.2. *Let D be a circle $|z| < R$, and let $M(r) = \max_{|z|=r} \|f(z)\|$, $0 \leq r < R$ where f is of type (A) on D to X. Then, if $0 \leq r_1 < r_2 < R$, $M(r)$ is strictly increasing on the interval $r_1 \leq r \leq r_2$ unless $\|f(z)\|$ is constant when $0 \leq |z| \leq r_2$.*

The theorem is an immediate corollary of Theorem 2.1. In certain applications it is very important to know that $M(r)$ is strictly increasing. In the classical theorem, where $X = Z$, $M(r)$ is strictly increasing unless f is constant. We cannot assert so much when the values of f are in a Banach space. It is possible to have a situation where f is analytic and not constant, but $\|f(z)\|$ is constant throughout an open set. This possibility has been noticed independently by various people. The following example, which is simpler than my own original one, was communicated to me orally by Max Zorn.

Example. Let X be the space of complex number pairs (w_1, w_2), with $\|(w_1, w_2)\|$ max $(|w_1|, |w_2|)$. Let $f(z)$ on Z to X be defined by $f(z) = (1, z)$.

Then $\|f(z)\| = 1$ if $|z| \leq 1$, and $\|f(z)\| = |z|$ if $|z| \geq 1$.

Theorem 2.3. *(The three circles theorem). Suppose that D is the annular region $0 \leq R_1 < |z| < R_2$, and that f is singlevalued and analytic on D to X; further suppose that $f(z)$ does not vanish identically. Then $\log M(r)$ is a convex function of $\log r$, $R_1 < r < R_2$ (where $M(r) = \max\limits_{|z|=r} \|f(z)\|$).*

The proof is exactly the same as in the classical case. See, for example, Titchmarsh [5], p. 172, and Landau [3], pp. 88-89.

3. *Banach spaces whose elements are members of $\mathfrak{U}$.* We begin with some statements about notation.

If $f \in \mathfrak{U}$ and x is a scalar parameter, the function $f(e^{ix}z)$ of the variable z is denoted by $U_x f$. It is a member of $\mathfrak{U}$.

If w is a complex parameter, with $|w| < 1$, the function $f(wz)$ of the variable z is denoted by $T_w f$. It is a member of $\mathfrak{U}$ if f is.

The function $z^n (n = 0, 1, 2, \ldots.)$ as a member of $\mathfrak{U}$ is denoted by u_n.

If $f \in \mathfrak{U}$ we write $a_n = a_n(f) = \dfrac{f^{(n)}(0)}{n!}$.

There are several subclasses of the class $\mathfrak{U}$ which are of great interest in analysis, and which may be regarded, under suitable definitions, as Banach spaces. In this paper we are interested in the space H^p (to be defined later), but it is convenient to develop our argument rather abstractly, so that we shall define a class of Banach spaces by selecting, as axioms, certain properties enjoyed by all those we have considered among the particular Banach spaces whose elements are members of $\mathfrak{U}$.

Definition 3.1. A subclass $\mathfrak{B}$ of $\mathfrak{U}$ will be called a normed linear space of type (T) if the following axioms are satisfied:

(1) $\mathfrak{B}$ is a linear space admitting complex numbers as scalars.
(2) To each $f \in \mathfrak{B}$ there corresponds a number denoted by $\|f\|$ such that with this as norm $\mathfrak{B}$ becomes a normed linear space.
(3) If $f \in \mathfrak{B}$ and x is real, then $U_x f \in \mathfrak{B}$ and $\|U_x f\| = \|f\|$.
(4) The elements u_n of $\mathfrak{U}$ belong to $\mathfrak{B}$, and their norms are bounded (say $\sup\limits_n \|u_n\| = M$).
(5) There exists an absolute constant N (depending only on $\mathfrak{B}$) such that $|a_n(f)| \leq N\|f\|$ when $f \in \mathfrak{B}$ and $n = 0, 1, \cdots$.

Definition 3.2. If $\mathfrak{B}$ is a normed linear space of type (T), and if it is complete, we shall call it a Banach space of type (T).

Theorem 3.1. *Let $\mathfrak{B}$ be a normed linear space of type (T). Then*

$$|f(z)| \leq \frac{N\|f\|}{1-|z|}$$

if $f \in \mathfrak{B}$ and $|z| < 1$.

The proof is an immediate consequence of axiom (5). Axioms (3) and (4) are not used in the proof.

Corollary. Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}$ (i.e. that $\|f_n - f_m\| \to 0$ as $m, n \to \infty$). Then $f_n(z)$ converges pointwise, when $|z| < 1$, to a function $f(z)$ belonging to $\mathfrak{A}$; the convergence is uniform in any region $|z| \leq r < 1$.

This follows at once from Theorem 3.1.

Theorem 3.2. Let $\mathcal{B}$ be a Banach space of type (T). Suppose $f \in \mathfrak{A}$ and $|w| < 1$. Then $T_w f \in \mathcal{B}$. As a function of w, $T_w f$ is analytic on $|w| < 1$ to $\mathcal{B}$, with the power series expansion

$$T_w f = \sum_{k=0}^{\infty} a_k w^k u_k.$$

Proof: The series $\sum_{k=0}^{\infty} a_k w^k u_k$ is certainly convergent in $\mathcal{B}$, since it is absolutely convergent and $\mathcal{B}$ is complete. The absolute convergence is a consequence of axioms (4) and (5).

It remains merely to show that the value of the series is the element $T_w f$ of $\mathfrak{A}$. Fixing w, let us write

$$g = \sum_{k=0}^{\infty} a_k w^k u_k, \qquad s_n = \sum_{k=0}^{n} a_k u_k.$$

Then

$$T_w s_n = \sum_{k=0}^{n} a_k w^k u_k,$$

and so $T_w s_n$ converges to g in $\mathcal{B}$. Hence it follows from Theorem 3.1 that $T_w s_n(z)$ converges pointwise to $g(z)$. On the other hand we have

$$T_w s_n(z) = \sum_{k=0}^{n} a_k w^k z^k,$$

and this converges pointwise to $T_w f(z) = f(wz)$. Consequently $g(z) = T_w f(z)$. This shows that $T_w f \in \mathcal{B}$ and completes the proof of the theorem. Observe that we did not use axiom (3) in the proof.

Theorem 3.3. Let $\mathcal{B}$ be a Banach space of type (T), and suppose $f \in \mathfrak{A}$. Then

(a) $\|T_w f\| = \|T_r f\|$ *if* $|w| = r < 1$.

(b) $\|T_r f\|$ *is a strictly increasing function of r on the interval $r_1 \leq r \leq r_2$, where $0 \leq r_1 < r_2 < 1$, unless $\|T_r f\|$ is constant on the interval $0 \leq r \leq r_2$.*

(c) *If $f(z) \neq 0$, $\log \|T_r f\|$ is a convex function of $\log r$ when $0 < r < 1$.*

Proof: We have $T_w f = U_x T_r f$ if $w = re^{ix}$; assertion (a) now follows from axiom (3). Thus $\max_{|w|=r} \|T_w f\| = \|T_r f\|$. Assertion (b) now follows from Theorems 2.2 and 3.2. If $f(z) \neq 0$, we infer from Theorem 3.1 that $\|T_r f\| > 0$ when $0 < r < 1$. Assertion (c) now follows from Theorems 2.3 and 3.2.

4. The space H^p.

Definition 4.1. By $H^p(p > 0)$ we mean the class of all elements $f \in \mathfrak{A}$ such that $\mathfrak{M}_p(f;r)$ is bounded as a function of r, $0 \leq r < 1$.

Definition 4.2. If $1 \leq p$ and $f \in H^p$ we write $\|f\| = \sup_{r<1} \mathfrak{M}_p(f;r)$.

Theorem 4.1. *With the norm defined above, $H^p(p \geq 1)$ is a Banach space of type* (T).

We shall leave to the reader the verification that H^p is a normed linear space of type (T). The constants M and N of axioms (4) and (5) are both unity in this case. The completeness of H^p is demonstrated as follows. Suppose that $\{f_n\}$ is a Cauchy sequence in H^p. By the corollary of Theorem 3.1, $f_n(z)$ converges pointwise to a limit $f(z) \in \mathfrak{A}$, the convergence being uniform in any region $|z| \leq r < 1$. Thus $\lim_{n \to \infty} \mathfrak{M}_p(f_n;r) = \mathfrak{M}_p(f;r)$ if $r < 1$. Now $\|f_n\|$ is bounded, say $\|f_n\| \leq A$, since $\{f_n\}$ is a Cauchy sequence. Hence $\mathfrak{M}_p(f;r) \leq A$. This proves that $f \in H^p$. If $\epsilon > 0$ is given, choose n_1 so that $m,n \geq n_1$ imply $\|f_n - f_m\| < \epsilon$. Then, if $r < 1$, $\mathfrak{M}_p(f_n - f_m;r) \leq \|f_n - f_m\| < \epsilon$. We may let $m \to \infty$ and obtain $\mathfrak{M}_p(f_n - f;r) \leq \epsilon$ if $n \geq n_1$. Since n_1 is independent of r, we have $\|f_n - f\| \leq \epsilon$ when $n \geq n_1$. Thus $f_n \to f$ in H^p, and H^p is complete.

Before considering H^p further, we prove an auxiliary theorem about real continuous functions.

Theorem 4.2. *Let $\varphi(x)$ be a real function which is defined and continuous when $0 \leq x < 1$ and such that $\varphi(0) \leq \varphi(x)$. Define*

$$\psi(x) = \sup_{0 \leq y < 1} \varphi(xy), \quad 0 \leq x < 1.$$

Suppose that ψ has the property that if $0 < x_1 < x_2 < 1$ and $\psi(x_1) = \psi(x_2)$, then $\psi(x)$ is constant on the interval $0 \leq x \leq x_2$. Then $\psi(x)$ and $\varphi(x)$ are identical.

Proof: From the definition and the continuity of φ we see that $\psi(0) = \varphi(0)$, $\psi(x) = \max_{0 \leq y \leq x} \varphi(y)$, $\varphi(x) \leq \psi(x)$. Suppose that for ξ we have $0 < \xi < 1$ and $\varphi(\xi) < \psi(\xi)$. Let x_1 be such that $0 \leq x_1 \leq \xi$ and $\varphi(x_1) = \max_{0 \leq x \leq \xi} \varphi(x) = \psi(\xi)$. Then $x_1 < \xi$. Also, $\psi(x_1) = \varphi(x_1)$. Thus, by the property assumed for ψ, $\psi(x)$ is constant when $0 \leq x \leq \xi$. Thus $\varphi(\xi) < \psi(\xi) = \psi(0) = \varphi(0)$. Since $\varphi(0) \leq \varphi(\xi)$, we have a contradiction and the theorem is proved.

Theorem 4.3. *Suppose $1 \leq p$, $f \in \mathfrak{A}$, $0 \leq r < 1$. Then*

$$\mathfrak{M}_p(f;r) = \|T_r f\|,$$

the norm being that of $T_r f$ as an element of H^p.

Proof: With p and f fixed, define $\varphi(r) = \mathfrak{M}_p(f;r)$. Clearly φ is continuous when $0 \leq r < 1$. Also, if $0 < r < 1$,

$$\varphi(0) = |f(0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})d\theta \right| \leq \mathfrak{M}_p(f;r) = \varphi(r),$$

by Cauchy's formula and Hölder's inequality. Next observe that

$$\mathfrak{M}_p(T_r f;\rho) = \mathfrak{M}_p(f;r\rho) = \varphi(r\rho),$$

whence

$$\|T_r f\| = \sup_{0 \leq \rho < 1} \varphi(r\rho).$$

Thus $\psi(r)$, as defined in Theorem 4.2, is $\|T_r f\|$ in the present instance. The property of ψ assumed in Theorem 4.2 is fulfilled because of Theorem 3.3 (b). Thus the conclusion of Theorem 4.3 is a particular instance of the conclusion of Theorem 4.2.

5. *Conclusion.* It follows from Theorem 3.3 and 4.3 that if $1 \leq p$ and $f \in \mathfrak{A}$, $\mathfrak{M}_p(f;r)$ is a nondecreasing function of r, $0 \leq r < 1$, and, if $f(z) \neq 0$, $\log \mathfrak{M}_p(f;r)$ is a convex function of $\log r$, $0 < r < 1$. Thus we have proved most of Theorem 1.1 for the case $1 \leq p$. The only thing remaining to prove is that $\mathfrak{M}_p(f;r)$ is a strictly increasing function of r when $f(z)$ is not a constant. The proof of this fact is based on the following result.

Theorem 5.1. Suppose $1 \leq p$, $0 < r < 1$, $f \in \mathfrak{A}$ and $f(z)$ not a constant. Then $\mathfrak{M}_p(f;0) < \mathfrak{M}_p(f;r)$.

This theorem shows that $\mathfrak{M}_p(f;r)$ is not constant in any interval $0 \leq r \leq \rho$. Hence, by Theorems 3.3 (b) and 4.3, $\mathfrak{M}_p(f;r)$ is a strictly increasing function of r when f is not a constant.

Proof of Theorem 5.1: Clearly $\mathfrak{M}_p(f;0) = |f(0)|$. Also, as is well known, $\mathfrak{M}_1(f;r) \leq \mathfrak{M}_p(f;r)$ when $p \geq 1$ (cf. [2], p. 143). Hence it suffices to prove that $|f(0)| < \mathfrak{M}_1(f;r)$. By Cauchy's integral formula,

$$(1) \qquad\qquad f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})d\theta,$$

whence $|f(0)| \leq \mathfrak{M}_1(f;r)$. Let us suppose that the equality holds here for some positive r, and hence for all sufficiently small r. It appears intuitively evident that the equality in question, namely

$$(2) \qquad\qquad \left| \int_0^{2\pi} f(re^{i\theta})d\theta \right| = \int_0^{2\pi} |f(re^{i\theta})|d\theta,$$

implies that the function values $f(re^{i\theta})$, for fixed r and varying θ, lie on a fixed ray emanating from the origin. (See Landau [3], p. 96, where the truth of this implication is accepted without demonstration.) We shall presently give a proof of the correctness of the implication. It is impossible for the function values $f(re^{i\theta})$ to lie on a fixed ray, however, for since f is analytic and not constant we know that as z goes once around a sufficiently small circle $|z| = r$ the point $w = f(z)$ follows a path which goes around the point $w = f(0)$ one or more times. Thus the equality (2) is impossible.

It remains to discuss the 'intuitively evident' conclusion to be drawn from (2). We offer one among various possible arguments in justification of the conclusion. With r fixed, write $f(re^{i\theta}) = u(\theta) + iv(\theta)$. Then (2) becomes

$$(3) \qquad \left(\int_0^{2\pi} u d\theta \right)^2 + \left(\int_0^{2\pi} v d\theta \right)^2 = \left(\int_0^{2\pi} (u^2 + v^2)^{1/2} d\theta \right)^2 .$$

Now, by Minkowski's inequality with index $\frac{1}{2}$,

$$(4) \qquad \left(\int_0^{2\pi} (u^2 + v^2)^{1/2} d\theta \right)^2 \geq \left(\int_0^{2\pi} |u| d\theta \right)^2 + \left(\int_0^{2\pi} |v| d\theta \right)^2 ,$$

equality occurring only if there exist real constants A, B, not both zero, such that $Au^2 = Bv^2$. (cf. [2], p. 146). Also,

$$(5) \qquad \int_0^{2\pi} |u| d\theta \geq \left| \int_0^{2\pi} u d\theta \right| ,$$

with a similar inequality for v. Moreover, since u is continuous, (5) is strict inequality unless u does not change sign. Thus we see that when (3) holds u and v are of constant sign and satisfy an identity $Au^2 = Bv^2$. Geometrically the conclusion is that the values of $u + iv$ lie on a fixed ray through the origin in the uv-plane.

We have now completed the proof of Theorem 5.1.

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Foreword. When a layman asks a mathematician "What is the fourth dimension?" he is usually quite surprised to be told that like the poor, the fourth dimension, and the fifth, and sixth, and n^{th} for that matter, "is with us always". And yet such is indeed the case. For the dimensionality of a collection of objects, i.e. a "space", is nothing but the number of independent data, or coordinates, which must be specified in order to identify uniquely a particular element of the space; and to insist that spaces of four or more dimensions are completely outside the bounds of experience is to deny that there are collections of familiar elements requiring four or more data for their specification.

In Relativity, to mention one example, the elements which are studied are events, and in as much as it takes three space coordinates to tell where, and one time coordinate to tell when a particular event occurred, the "space" of events is four dimensional, although all its elements have as real an existence as any other objects of scientific inquiry. Similarly, if one considers all the conics which can be drawn in a given plane, real as they are they nonetheless form a five dimensional collection or "space", since five data, for instance the ratios of any five of the coefficients in the general equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

to the sixth, are required to identify a particular conic.

As a practical detail in studying collections like these, mathematicians usually think of the elements as "points", and invent modes of representation or "visualization" which permit such constructs as lines, planes, conics, and quadric surfaces to be generalized to objects possessing analogous properties, which, after a little practice, can be handled almost as easily as their counterparts in two and three dimensions.

A classical application of this point of view is to the study of families of lines in ordinary three-space, S_3. In such a space let a homogeneous, or projective coordinate system be established, and let

$$Q:(x_1,x_2,x_3,x_4) \quad \text{and} \quad R:(y_1,y_2,y_3,y_4)$$

be any two points on a general line, l. Then in a certain sense l is characterized by the six second order determinants

$$P_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

which can be derived from the matrix

$$\left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right\|$$

125

In fact it can easily be shown that if two new points, Q' and R', are chosen on l, the new determinants, P'_{ij}, are always proportional to the old. Moreover by expanding the null determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$$

in terms of the minors of the first two rows, one obtains the identity

1) $$P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23} = 0$$

Thus any six numbers satisfying the quadratic identity, 1), can be regarded as the coordinates of some line, and all such sets which are proportional are associated with the same line. In the literature of line geometry the P_{ij} are known as Plücker line coordinates. Since the coordinates of a line must always satisfy the identity 1), and since only the ratios of the P_{ij} are significant, it is evident that of the six Plücker coordinates only four are essential. Hence the lines of S_3 form a four dimensional collection or "space".

If the P_{ij} be regarded as the homogeneous coordinates of a point in a mapping, or representation space of five dimensions, then 1) can be regarded as the equation of a generalized or "hyper"-quadric surface. Any six numbers, P_{ij}, which are the coordinates of a line in S_3 are then the coordinates of some point on this hyperquadric, which is usually denoted by the symbol V_4^2, and conversely any point on V_4^2 has coordinates P_{ij} which are associated with a unique line in S_3. Any property of the lines of S_3 can thus be interpreted as a property of the points of V_4^2, and any property of the points of V_4^2 implies some property of the lines in S_3.

In the present paper we are concerned with involutorial line transformations in S_3, that is with transformations which send a line l into a line l', and which, when repeated, send l' back into l. However, instead of studying such transformations in S_3 we find it more convenient to consider them as point transformations which send a point P of V_4^2 into another point, P' of V_4^2, and which when repeated, send P' back into P. After defining four simple numerical characteristics of line involutions we raise the question "Do involutions exist corresponding to all possible sets of these characteristics?" Known involutions by no means provide examples of all possible types, and the main result of our paper is the description of a new series which, with previously known examples, shows the existence of involutions associated with all possible sets of characteristics except those in which the lines left invariant by the transformation form a linear complex, i.e. are represented by the points common to V_4^2 and some "hyperplane" of S_5. Further study has already revealed examples of some of these missing involutions, and there is every reason to expect that representative

involutions will be found for all sets of characteristics. However these results are not discussed in this paper.

I. It is well-known that line geometry in S_3 is completely equivalent to point geometry on a non-singular V_4^2 in S_5. From this point of view, involutorial line transformations in S_3 can equally well be studied as involutorial point transformations of the fundamental V_4^2 into itself. Such transformations evidently fall into one or the other of two great classes:

a) Those for which the line joining a general point, P, to its image, P', lies entirely on V_4^2, and

b) Those for which this is not the case.

Involutions of the first type, which constitute the subject matter of this note, possess four important numerical characteristics. If l be a general line of V_4^2, these may be defined as

a) The order, m, of the curve of images of the points of l, that is, the order of the involution itself.

b) The number, i, of points of l which coincide with their images, that is, the order of the complex of invariant elements.

c) The order, n, of the ruled surface formed by the lines which join the points of l to their respective images.

d) The number, k, of points of l with the property that the lines joining them to their images lie entirely on V_4^2. In S_3, k is the number of lines of an arbitrary pencil which meet their images.

The four quantities m, i, n, k are not independent. In fact they satisfy the following identities:

A) $m + 1 = 2n - k = n + i$

These can easily be established by considering the ruled surface whose generators join the points of l to their respective images. From the nature of the intersection of this surface with V_4^2 and with a general S_4 through l, the relations A can be obtained at once.

It is a matter of some interest to know whether a given set m, i, n, k, is adequate to characterize a unique involution. This is sometimes the case, for it is an easy matter to show that the involution defined by a linear line complex is uniquely determined by the characteristics $(1,1,1,0)$. Moreover in a recent paper[1] we have shown that there is but one involution associated with the set $(2,1,2,1)$, namely the transformation of V_4^2 into itself effected by the transversals

of a general line of S_5 and one of its semi-polar $S_3's$ as to V_4^2. In general such unique determination does not obtain, however.

A second question of importance is whether or not there exist involutions corresponding to every set of non-negative integers satisfying A. The literature of algebraic geometry contains many examples of line involutions, including some series of infinitely many related involutions.* These include instances corresponding to all sets of characteristics for which $m < 4$, but by no means all for which $m \geq 4$. It seems appropriate, therefore, to call attention to a new class of involutions providing examples of all orders $m \gtreqless 4$ for which i can range from 2 up to its maximum value, namely the largest integer $\lesseqgtr \dfrac{m + 1}{2}$.

The series of involutions discussed by Clarkson (3) furnishes examples of all orders for which $i = 0$, hence the question of existence is now reduced to involutions for which $i = 1$. Although only isolated examples for which $i = 1$ have been discussed, there is no reason to expect that involutions of all orders having $i = 1$ will not be revealed by further study.

II. The new involutions we define in S_5 as follows. Let there be given a rational curve, C_r, and an S_3, Σ, having the minimum possible intersection with the subspace in which C_r is contained, and meeting C_r in $(r - 1)$ points. Through a general point, P, on V_4^2 there passes a unique line incident to C_r and Σ at points not common to the two. The second intersection of this line with V_4^2 is the image, P', of P. In three dimensions this definition becomes the following. Let there be given

a) a bilinear congruence, and
b) a one-parameter, non-linear family of order r of linear complexes with the property that $(r - 1)$ of the complexes contain the directrices of the congruence.

A general line, l, and the given congruence determine a linear complex which is in involution with exactly r of the complexes of the given family, including the $(r - 1)$ which contain the directrices of the congruence. The transform of l by the remaining complex is the image, l', of l.

To determine the characteristics of these involutions we observe first that the order, n, of the regulus of lines which join the points of a general line, l, to their respective images is $(r + 1)$. In general none of these rays will lie on V_4^2, hence $k = 0$. Equations A then give at once

$$m = 2r + 1 \quad \text{and} \quad i = r + 1$$

* A partial bibliography of such results is given at the end of this paper.

In S_5 the invariant complex consists of a one-parameter, non-linear family of quadric surfaces. For any invariant point is necessarily the point of contact of a tangent to V_4^2 which meets C_r and Σ, and all such points will lie on the quadric surfaces which are common to V_4^2, the polar $S_4's$ of the continuum of points, Q_j, of C_r, and the $S_4's$: (Σ, Q_j). In three dimensions the invariant lines thus form a complex of order $(r + 1)$ consisting of ∞' bilinear congruences whose directrices form a ruled surface of order $2(r + 1)$.

In S_5 the singular elements are of three types:

a) The points of the quadric surface in which Σ meets V_4^2, each of which is transformed into the curve or order $2r$ which is the intersection of V_4^2 and the conical surface having the point in question for vertex and C_r for directrix.

b) The points of the $2r$ quadric cones having as vertices the points in which C_r meets V_4^2, and as directrices the conics in which Σ is met by the V_3^2's cut from V_4^2 by its tangent S_4's at these intersections. Such points are all transformed into the entire generators on which they lie.

c) The $2r$ points in which C_r meets V_4^2, each of which is transformed into the V_3^2 which is the intersection of V_4^2 and the S_4 determined by Σ and the point in question.

In three dimensions there is thus one bilinear congruence each of whose lines is transformed into a ruled surface of order $2r$, $2r$ special bilinear congruences each of whose lines is transformed into a plane pencil, and $2r$ lines each of which is transformed into an entire linear complex.

The class of the image congruence of a plane field of lines represented on V_4^2 by a plane, π, will be the number of transversals of C_r and π which meet an arbitrary plane, π', of the same family as π. All such transversals must lie in the S_4: (π, π'). This meets Σ in a plane and C_r in r points. Since in S_4 a unique line can be drawn through a given point to meet three given planes, the class of the image congruence is r.

The order of the image congruence will be the number of transversals of π, C_r, and Σ which meet an arbitrary plane, σ, of the family opposite to π. Consider therefore the S_4 determined by σ and any line, l, in π. This meets C_r in r points, through each of which passes a trans-

versal of σ, C_r, and l. There is thus in π a curve, Γ_r, of order r through each point of which passes a transversal of σ and C_r. The ruled surface formed by the joins of corresponding points on C_r and Γ_r is of order $2r$. This surface is met by Σ in $2r$ points, from which must be deducted the $(r - 1)$ points common to Σ and C_r, leaving $(r + 1)$ as the required order. By identical reasoning the image congruence of a bundle of lines can be shown to be of class $(r + 1)$ and order r.

III. For i to assume values less than $(r + 1)$ it is necessary that the involution be specialized in such a way that $k > 0$. We may readily make $k = 1$ or 2 (and therefore $i = r$ or $r - 1$) by constructing C_r to pass through the necessary $(r - 1)$ points in Σ and through one or both of the intersections of V_4^2 and the polar line of Σ.

For k to be greater than 2, with consequent reduction of m and i, it follows that there must be more than two points common to C_r and V_4^2 at which the tangent S_4 contains Σ. But all such points must lie on the polar line of Σ. Hence if k is to exceed 2 it is necessary that the polar line of Σ meet V_4^2 in more than two points, that is, that it must lie entirely on V_4^2. For this to happen, the intersection of Σ and V_4^2 must reduce to a pair of planes, their intersection being the polar line of Σ which C_r is to meet more than twice.

This condition is not sufficient, however, for the unique point on a line, l, of V_4^2 whose image is one of the intersections, Q, of of Σ and C_r is the intersection of l and the S_4 determined by Σ and the tangent line to C_r at Q, and this point will not in general coincide with the point of intersection of l and the tangent S_4^- to V_4^2 at Q. The complete condition is therefore that Σ be the polar of a line of V_4^2 which is met by C_r in more than two points, and that at each of these points the S_4 determined by Σ and the tangent to C_r be the tangent S_4 to V_4^2. This evidently rules out the case in which C_r is a plane curve, for then C_r must have a single $(r - 1)$-fold point in common with Σ and no others. If C_r is a space curve it must have an $(r - 1)$-fold secant in common with Σ. If this line be taken on V_4^2, and if Σ be taken as the polar of this line, and if the tangent to C_r at each of these points be tangent also to V_4^2, the above conditions will be fulfilled and k may be as much as but no more than $(r - 1)$, implying that i may be as small as but no less than 2. If

C_r lies properly in an S_4 no more than $(r - 2)$ of the $(r - 1)$ points which it must have in common with Σ can be collinear. The upper limit for k is then $(r - 2)$. If C_r lies properly in five dimensions, the maximum value of k is only $(r - 3)$.

The existince of involutions of these special types depends of course upon the existence of curves bearing the appropriate special relations to V_4^2. Such curves do exist and in fact can be obtained by projecting the rational normal r-ic onto an S_3 from the S_{r-4} in which two of its $(r - 1)$-secant S_{r-2}'s intersect, or by projecting onto an S_4 or an S_5 from a suitably chosen S_{r-5} or S_{r-6} in such an S_{r-4}. The projected curves in each case will lie on quadric primals, proper when the projection is onto an S_3, conical in the other two cases, and will have the desired multiple secants. Choosing V_4^2 in each case to be a non-singular quadric primal tangent along the multiple secant to the quadric bearing the projected curve, the requisite configuration is obtained.

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ON A FORMULA FOR CIRCULAR PERMUTATIONS

Harold S. Grant

In the solution of Problem E 678, American Mathematical Monthly, June-July, 1945, the following formula was derived for the number P of circular permutations of $N = \sum_{i=1}^{n} x_i$ letters of which x_1 are alike, say A_1, x_2 others alike, say A_2, $\cdots$, x_n others alike, say A_n:

$$(1) \qquad P = \frac{1}{N} \sum_{d/D} \varphi(d) \left[\frac{(N/d)!}{\prod_{i=1}^{n} (x_i/d)!} \right],$$

where $D = (x_1, x_2, \cdots, x_n)$, the greatest common divisor of the x_i's, $\varphi(d)$ is Euler's φ-function, and the summation is to be taken over all divisors d of D. Particular cases of this problem have occurred in various places (e.g. Problem 519 in the April 1944 issue of the National Mathematics Magazine, and Problem E 599 (1943, 634) in the American Mathematical Monthly for October 1944, pp: 472-3).

In the "Proceedings of the Edinburgh Mathematical Society, Vol: 8, 1890, pp: 64-69", there appeared an article by R. E. Allardice "On a problem in permutations", viz: "How many necklaces may be formed with p pearls, r rubies, and d diamonds?" This problem had been suggested by Professor Chrystal, and was accompanied with the remark that the general solution cannot be given in terms of p, r, and d alone, but depends on the nature of the numbers p, r, and d. In view of the formula just given, we wish now to show that this remark is not entirely correct.

In precisely the same way as we define a circular permutation as "one that is invariant under a cyclic interchange of the letters", we define a "necklace" permutation as "a circular permutation invariant under a reversal of the letters". From this definition, it is apparent that the number of "necklace" permutations will give the actual number of "necklaces". To obtain the number of "necklace" arrangements of the A_i's, we note that, if among the P circular permutations there are N_1 which remain the same under a reversal of the letters, then the number of "necklaces" is equal to $N_1 + (P - N_1)/2 = (P + N_1)/2$. Consequently, we seek the N_1 circular permutations mentioned.

Considering such a permutation, we note that it must be such that, when read clock-wise from some A_i on a circle, we must reach an A_i from which the permutation is unchanged reading counter clock-wise. The

letters A_j, therefore, may be arranged symmetrically about a diameter, so that, if x_j is odd, A_j must come at the end of this diameter. We thus have three, and only three, cases in which any circular permutations of the prescribed type could occur, viz: (1). One odd, x_j say, in which one end of the diameter is occupied by A_j. (2). Two odd, x_j and x_k say, in which both ends of the diameter are occupied by A_j and A_k. (3). No odd, in which case both ends of the diameter may or may not be occupied.

In the first case, the letters A_i arrange themselves symmetrically on either side of A_j, as, e.g. $A_1 A_2 A_3 \cdots A_{j-1} A_j A_{j-1} \cdots A_1$, $j > 1$. We wish now to show that there are as many different circular arrangements of this type as there are ways of arranging $A_1 A_2 A_3 \cdots A_{j-1}$, i.e. $(\frac{N-1}{2})! / (\frac{x_1}{2})! (\frac{x_2}{2})! \cdots (\frac{x_j-1}{2})! \cdots (\frac{x_n}{2})!$. This is readily verified for $j = 2$. To complete the proof by induction, we note that the circular arrangement $A_1 A_2 \cdots A_{j-1} A_j A_{j-1} \cdots A_1$, $j > 1$, could give rise to another arrangement of the prescribed type only when two adjacent A_i's are the same. Now if $j > 2$ and $A_i = A_{i-1}$, $1 < i < j$, then $A_i A_i$ occurs on either side of the fixed element A_j, and may be replaced by a single letter, thus affording a reduction in the number of letters. If $A_{j-1} = A_j$, then $A_{j-1} = A_{j-2}$ for symmetry about $A_{j-1} A_j$ or $A_j A_{j-1}$, again affording a reduction in the number of letters. Thus, whenever the conditions are present for possible duplication, the number of letters may be reduced, which completes the proof by induction. The case $j = 1$, i.e. the case of a single letter, is readily seen to come under the same formula, since $0! = 1$.

In the second case, the letters arrange themselves symmetrically on either side of two fixed letters A_j and A_k ($A_j \neq A_k$), as e.g. $A_1 A_2 \cdots A_{k-1} A_k A_{k-1} \cdots A_1 A_j$, $k > 1$. For each arrangement of $A_1 A_2 \cdots A_{k-1}$ there corresponds one and only one circular arrangement symmetric about A_k, and hence one and only one of the prescribed type. This gives just $(\frac{N-2}{2})! / (\frac{x_1}{2})! (\frac{x_2}{2})! \cdots (\frac{x_j-1}{2})! (\frac{x_k-1}{2})! \cdots (\frac{x_n}{2})!$ circular permutations of the prescribed type. To give a rigorous inductive proof of this case, we will find it convenient to make a slight change in notation, and write the circular permutation involved as $A_1 A_2 A_3 \cdots A_{j-1} A_j A_{j-1} \cdots A_2$, $j > 2$, $A_j \neq A_1$. What we wish to show is that, when we associate with each arrangement of $A_2 A_3 \cdots A_{j-1}$ the circular permutations $A_1 A_2 A_3 \cdots A_{j-1} A_j A_{j-1} \cdots A_2$ and $A_j A_2 A_3 \cdots A_{j-1} A_1 A_{j-1} \cdots A_2$, each circular permutation will admit but one duplicate of the prescribed type, namely $A_j A_{j-1} \cdots A_2 A_1 A_2 \cdots A_{j-1}$ and $A_1 A_{j-1} \cdots A_2 A_j A_2 \cdots A_{j-1}$, respectively. The result will then follow. We verify the cases $j = 3$ and $j = 4$ directly. The case $j = 3$ offers no difficulty. For $j = 4$ we have $A_1 A_2 A_3 A_4 A_3 A_2$. To give rise to a duplicate other than $A_4 A_3 A_2 A_1 A_2 A_3$, we must have symmetry about A_2 or A_3 either of which implies $A_3 = A_1$, $A_4 = A_2$, giving

$A_1A_2A_1A_2A_1A_2$, which possesses but the one duplicate $A_2A_1A_2A_1A_2A_1$. For $j > 4$, symmetry about A_i, $2 < i < j-1$, implies that $A_{i-1}A_iA_{i-1}$ occurs on either side of the fixed letter A_j, and may be replaced by a single letter, affording a reduction in the number of letters. Symmetry about A_2 implies $A_4 = A_2$, so that $A_2A_3A_2$ occurs on either side of A_j. Finally, symmetry about A_{j-1} implies $A_{j-1} = A_{j-3}$, so that $A_{j-3}A_{j-2}A_{j-3}$ occurs on either side of A_j. Thus, for those cases in which more than one duplicate may be present, the number of letters may be reduced, which completes the proof by induction. The case $j = 2$, i.e. the case of just two different letters, is readily seen to come under the same formula.

In the third case, we may have either $A_1A_2A_3 \cdots A_{m-1}A_mA_mA_{m-1} \cdots A_1$, which we will call type 1, or $A_1A_2A_3 \cdots A_{m-1}A_mA_1A_mA_{m-1} \cdots A_2$, which we will call type 2. We define type 2 as A_1A_1 for $m = 1$. We wish to show that, when we associate with each arrangement of $A_1A_2 \cdots A_m$ the circular permutations $A_1A_2 \cdots A_mA_m \cdots A_2A_1$ and $A_1A_2 \cdots A_mA_1A_m \cdots A_2$, each circular permutation will admit but one duplicate of the prescribed types. This is readily verified for $m = 1$ and 2. For $m > 2$, confining our attention first to type 1, we note that if $A_i = A_{i-1}$, $1 < i \leq m$, A_iA_i would occur on either side of the center, and the number of letters could be reduced. Symmetry about A_1 would imply $A_2 = A_1$; symmetry about A_m would imply $A_{m-1} = A_m$; symmetry about A_i, $1 < i < m$, would imply the occurrence of like combinations $A_{i-1}A_iA_{i-1}$ on either side of the center. In all of these cases, therefore, the number of letters may be reduced, and these comprise the only cases in which the circular permutation $A_1A_2 \cdots A_mA_m \cdots A_2A_1$ could possess any duplicate of the prescribed types other than $A_m \cdots A_2A_1A_1A_2 \cdots A_m$.

Turning our attention now to type 2 for $m > 2$, we note that if $A_i = A_{i-1}$, $2 < i \leq m$, A_iA_i would occur on either side of the center. If $A_1 = A_2$, symmetry about A_1A_2 would imply $A_2 = A_3$. If $A_m = A_1$, symmetry about A_mA_1 implies $A_m = A_{m-1}$. For $m > 3$, symmetry about A_i, $2 < i < m$, implies the occurrence of $A_{i-1}A_iA_{i-1}$ on either side of the center. Symmetry about A_2 implies $A_4 = A_2$, so that $A_2A_3A_2$ occurs on either side of the center; symmetry about A_m implies $A_m = A_{m-2}$, so that $A_{m-2}A_{m-1}A_{m-2}$ occurs on either side of the center. For $m = 3$, symmetry about either A_2 or A_3 would imply $A_1 = A_2 = A_3$. Thus, in all cases in which the circular permutation $A_1A_2 \cdots A_mA_1A_m \cdots A_2$ could possess any duplicate of the prescribed types other than $A_1A_m \cdots A_2A_1A_2 \cdots A_m$, the number of letters may be reduced. This completes the proof of this case by induction, and the reader should note that the proof implies that there may be a correspondence between types, rather than within types, only when the circular permutation may be reduced to A_1A_1. The equality of any of the three pairs of linear permutations, viz:, either $A_1A_2 \cdots A_m$ and $A_m \cdots A_2A_1$, or $A_1A_2 \cdots A_m$ and $A_1A_m \cdots A_2$, or $A_m \cdots A_2A_1$ and $A_1A_m \cdots A_2$, leads to essentially this case.

We tabulate the arrangements and types for the case of 4A's and 2B's and 6A's and 4B's. Those linear arrangements in which one of the two corresponding circular permutations reduces to A_1A_1 have been noted, and the duplicates indicated.

4 A's; 2 B's

Arrangements	Type 1	Type 2
→ ABA	ABAABA ←	ABAAAB ⌇
AAB	AABBAA ⌇	AABABA ⌇
→ BAA	BAAAAB ⌇	BAABAA

6 A's; 4 B's

Arrangements	Type 1	Type 2
AAABB	AAABBBBAAA ⌇	AAABBABBAA ⌇
BBAAA	BBAAAAAABB ⌇	BBAAABAAAB ⌇
→ BAAAB	BAAABBAAAB ⌇	BAAABBBAAA ⌇
BABAA	BABAAAABAB ⌇	BABAABAABA ⌇
BAABA	BAABAABAAB ⌇	BAABABABAA ⌇
→ ABAAB	ABAABBAABA ⌇	ABAABABAAB
AABAB	AABABBABAA ←	AABABABABA ⌇
→ AABBA	AABBAABBAA ⌇	AABBAAABBA
→ ABABA	ABABAABABA ⌇	ABABAAABAB ⌇
ABBAA	ABBAAAABBA ←	ABBAAAAABB ⌇

Thus, in the third case, we have just $(\frac{N}{2})!/(\frac{x_1}{2})!(\frac{x_2}{2})!\cdots(\frac{x_n}{2})!$ circular permutations of the prescribed type.

Defining the modified Kronecker delta $\delta^x_{abc\cdots}$ as 1 when x is either a, b, c, $\cdots$, and as 0 otherwise (including the case in which there is no subscript), we may summarize our findings in the following theorem: The number of "necklace" permutations of $N = \sum_{i=1}^n x_i$ letters of which x_1 are alike, say A_1, x_2 others alike, say A_2, $\cdots$, x_n others alike, say A_n is given by the formula:

$$P/2 + \delta^x_{012}\left(\frac{N-\int_1^x - 2\int_2^x}{2}\right)\bigg|\bigg/ 2\prod_{i=1}^n\left(\frac{x_i - \int_{jkl\cdots}^i}{2}\right)!,$$

in which x is the number of odd x_i, j, k, 1, $\cdots$ are the subscripts for the odd x_i, and P is given by (1). P has the same parity always as

$$\delta^x_{012}\left(\frac{N-\int_1^x - 2\int_2^x \cdot}{2}\right)!\bigg/\prod_{i=1}^n\left(\frac{x_i - \int_{jkl\cdots}^i}{2}\right)!.$$

Rutgers University

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Edited by

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This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topis related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

SOME TYPES OF ELEMENTARY EQUATIONS

Carl H. Denbow

When a student of algebra sees an equation like $5x + 7x = 12$ he is expected to respond by solving it; an equation like $5x + 7y = 12$, on the other hand, is supposed to evoke his graphing instincts; while equations like $5x + 7x = 12x$ and $x^2 - xy = x(x - y)$ should call forth neither of these responses, but rather a feeling of approval, of agreement.

These three kinds of responses arise from the fact that a large portion of elementary mathematics centers around problems of three kinds.* The first problem is that of finding and using universal number-truths, or *identities*, such as the fact that "any number added to itself gives twice that number", or, as it is usually written, $x + x = 2x$. Again, the fact that "two times any first number plus two times any second number is equal to two times the sum of those numbers", generally written $2x + 2y = 2(x + y)$, illustrates an identity in both x and y. The second problem is that of finding and handling useful *relations between numbers*. Thus the equation $5x = 9y + 160$ relates x and y so that x is Fahrenheit temperature when y is Centigrade temperature, but this equation is not an identity since it is not true for numbers x and y selected at random. As another example, $x^2 + y^2 = 1$ is true only for those numbers which, in an xy coordinate system, represent points on the unit circle. The third problem is that of finding a *number (or numbers) with certain properties*. Thus to find a number whose square exceeds the number by two, we generally solve the equation $x^2 - x - 2 = 0$.

It hardly needs to be mentioned that even those students who generally react "correctly" to equations of the three types illustrated above are often not conscious of the reasons for doing so; and hence fail to react correctly if the same equations are met unexpectedly or in a strange environment. For example, if asked to compute $438^2 - 413^2$, their direct computation shows that they do not really believe that $x^2 - y^2 = (x - y)(x + y)$ holds for all numbers, including 438 and 413.

*In what follows, x and y, unless otherwise restricted, will range over all real numbers.

They do not realize that such equations as $xy + xy = 2xy$ or $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ owe their importance and usefulness to the fact that they really work for all numbers. They are sometimes led to strange results by confusing unknowns with variables.

Any remedy for this situation must require, among other things, a better understanding of various types of equations. Unfortunately, however, a rigorous classification demands a mathematical maturity far beyond that generally expected of, say, a college freshman, since the sophisticated mathematician would require; first, a specification of a number-system, such as a domain of integrity, ring, field, etc.; second, that a function or set of functions $f(x,y)$, defined on such a number-system or subset and with specified properties, be given; then he would inquire into the nature of the solutions of $f(x,y) = 0$. Such an approach is clearly unsuitable for beginners. However, the thesis developed in this article is that a very elementary classification is possible on a pre-college plane which can be refined and sharpened in each later course. This procedure will benefit the student in several ways. The conscious effort to classify an equation forces the student to think about some crucial mathematical questions. It serves as a corrective for the fallacy that equations are self-explanatory and of unlimited applicability. Too many students learn by rote memory how to do such things as divide algebraic expressions, to solve quadratic equations, to substitute in formulas, so that there is need for a teaching procedure which will show some of the inter-relations of these processes.

To illustrate these remarks, let us outline a method of teaching some ideas about equations to rather immature students, and then discuss some of the refinements which should be added later. Let us begin with the truism that much of mathematics is stated in, and deals with, equations involving letters. "Now it is a grave mistake", (we might say), "to think that these letters always refer to unknown numbers. In fact, equations to be solved for unknowns become less and less important relatively in later mathematical and engineering work. Instead, the purpose of many equations is to state truths about all numbers. Such equations are called identities. When you see an identity you may handle it quite differently from other types of equations; hence it is important to recognize the type. For example, you have learned how to add such expressions as $2x + 3y$ and $x - 5y$; how to multiply them; how to factor such expressions as $x^2 - 5xy + 4y^2$. It is essential to your mathematical development that you should not be so absorbed with these *processes* of adding, multiplying, etc., that you fail to grasp the importance of the *results*. The first step in understanding the results is to write them as equations. Thus, our illustrations above will be written

$$(2x + 3y) + (x - 5y) = 3x - 2y,$$
$$(2x + 3y)(x - 5y) = 2x^2 - 7xy - 15y^2,$$

and $$x^2 - 5xy + 4y^2 = (x - 4y)(x - y).$$

Each of these equations is an identity, i.e., it works for all numbers x and y whatsoever. Remember that these equations are important, not because they tell how to multiply and add, etc., but rather the processes of adding and multiplying algebraic expressions are important as means to the end of finding identities. Begin now to make a list of the processes which lead to identities and of the things you do and don't do with identities." The list of processes applicable to identities will come to include such items as: a) One does not try to graph or solve identities; b) an identity in x and y becomes an identity in a and b if x is replaced by $a + b^2$ and y by $a^2 b - 3a$, for example; c) an identity in x remains true if x is replaced by $2x$, or by $x^2 - 8$, say, in striking contrast with equations in the "unknown x" like $x - 2 = 10$; d) if identities are added or multiplied, the resulting equations are again identities. There is no need to extend this list, as each reader will have his own way of organizing facts about identities.

At an appropriate time the study of different types of equations would be continued by showing the student that most of the equations of applied mathematics and many in pure mathematics are not identities, but are equations in two or more letters (which are now known as *variables*) called conditional equations, or equations of relation. Formulas are of this type, and equations of curves, and equations which determine one letter as a function of another. In a sense an equation of relation (in x and y, say) is opposite in nature to an identity, for instead of being true for all numbers its peculiar merit is that it *relates* the numbers x and y and thus *restricts* the possible values of the number pair (x,y). When we graph $y = x^2$, for example, we see that it is useful because it restricts the numbers x,y to pairs such as $(1,1)$ or $(2,4)$ which lie on a parabola. Much of analytic geometry is incomprehensible unless this idea of restriction is given a central place. The graph of $xy = yx$, which does not restrict x and y, is the whole xy plane; hence graphing is of little or no value when applied to identities. Again, an equation of relation between variables is peculiarly suited to problems involving continuous change, and hence is likely to be used in calculus. Many other items would enter the student's notebook regarding equations of relation, such as the fact that a formula like $5F = 9C + 160$ yields an equation involving an unknown when one of the variables F and C is given a numerical value; this statement does not apply to an identity. Each reader can supply many other facts of this nature.

The brief and sketchy discussion above is perhaps sufficient to illustrate a method of teaching identities and equations of relation. Let us add a few words regarding equations with unknowns. This type is sometimes taught first, and often receives undue emphasis. It should, however, be noted that the solution of many equations with unknowns

involves the use of identities. Thus, $2x - 3 = 5x + 12$ may be solved by adding to both sides the identity $-2x - 12 = -2x - 12$. A less trivial example is the solution of $x^2 - x - 2 = 0$, say, by factoring. This method is made possible by the fact that the identity $x^2 - x - 2 = (x - 2)(x + 1)$ is true for every number x, hence it must be applicable to the unknown number for which we are searching.

Although the preceding remarks have been very incomplete, they will serve as a background for discussing the interesting and crucial question of rigor in teaching the classification of equations. The three types of equations referred to above were introduced by examples, and not by rigorous definitions. In fact, due partly to the presence of many "borderline" cases, it seems unwise to give rigorous definitions to first or (possibly) second year college students. However, many of these borderline cases can and should be introduced, even in secondary school, to provide the incentive for sharper analysis and clearer concepts. As somewhat random illustrations, an equation like $1/x + 1/y = (x + y)/xy$ is an identity except when x or y is zero; $x^2 + y^2 = 1$ is an equation of relation between real numbers only when x and y are numbers between -1 and 1, inclusive; the logarithmic identity $\log xy = \log x + \log y$ holds, in the domain of real numbers, only when x and y are positive. The equation $x = 2$ can be (and, in plane analytic geometry, is) considered as $x = 2 + 0 \cdot y$, which is an identity in y for $x = 2$, i.e., it is true for all points $(2, y)$. More difficult cases may be met, in later courses, of equations which are identities on one range, but are equations of relation or perhaps meaningless on some other ranges. Again, the equation $x^2 + 2y = 5$ is usually considered as an equation of relation, but one can think of it as defining the function $y = (5 - x^2)/2$; and if y were thought of, not as a variable, but as a symbol for the function $(5 - x^2)/2$ then the equation $x^2 + 2y = 5$ would be an identity in x. The equation $1/(1 - x) = 1 + x + x^2 + \cdots$ is an identity for $-1 < x < 1$, but the infinite series on the right diverges for other values of x. As a final illustration, $xy = yx$ is an identity for real and complex numbers, but not for matrices or vector cross products.

It is possible to draw widely divergent conclusions from such examples as those just listed. It may be argued that they show the futility of trying to teach high school students, or beginning college students, the differences between identities and equations of relation and equations with unknowns; that the point of view presented earlier in this paper is such a gross oversimplification that it is practically worthless. On the other hand, one may decide from these and similar examples that the task of classifying equations is so fundamental and has so many ramifications that it should be undertaken early, so that a background will be developed before some of the difficulties are encountered.* This will probably be the conclusion of one who believes,

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*It may not be out of place to state here a more radical application of these

as does the writer, that without some help in analyzing the different
types of equations with which he must work the average student will fall
back on a memorized, mechanical approach to mathematics which will
sharply limit his ability to pursue the subject further. It is gener-
ally agreed that many of the basic ideas of mathematics, such as
function, limit, and so on, should be introduced as early as is
practicable, and repeated frequently. Perhaps a similar treatment of
some of the questions relating to equations would help the beginning
student to see mathematics as a coherent system of ideas.

ideas, which the writer believes worth trying. Namely, algebra would be intro-
duced early in grade school in the form of easy identities written in words,
such as "any number added to itself gives twice that number". Not until a dozen
or so such statements had been thoroughly assimilated would they be abbreviated
into $n + n = 2n$, etc., which would still be *read in words*, just like the English
statements. Equations involving unknowns would not be used until much later
Furthermore, a uniform notation would be adopted and until a certain stage all
equations found in the text or written by the instructor or pupil would be so
written as to indicate whether they are identities, formulas, or equations with
unknowns. This notation might involve different kinds of type, or the reserving
of certain letters for each kind of equation, or other expedients. This arti-
ficial aid in classifying equations would be discarded in senior high school or
early in college, and the student should by then be prepared to make his own
analyses.

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METRIC DIFFERENTIAL GEOMETRY
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You recall with what delight you first discovered that many geometrical and physical situations can be analyzed, and problems solved, by means of differential and integral calculus. Fortunately this pleasure of discovery can be continued, for many parts of more advanced mathematics and physics use the concepts of calculus as basic tools. This is particularly true in the study of properties of curves and surfaces in space.

Just as in plane analytic geometry we ordinarily use equal scales on two directed axes at right angles to each other in locating points (x,y), in solid analytic geometry we use equal scales on three concurrent directed axes which are at right angles each to each in locating points (x,y,z). It is clear that if two of these directions are fixed, then there are exactly two possible directions for the third, as in Figures 1 and 2. In the first of these figures, an ordinary right-hand
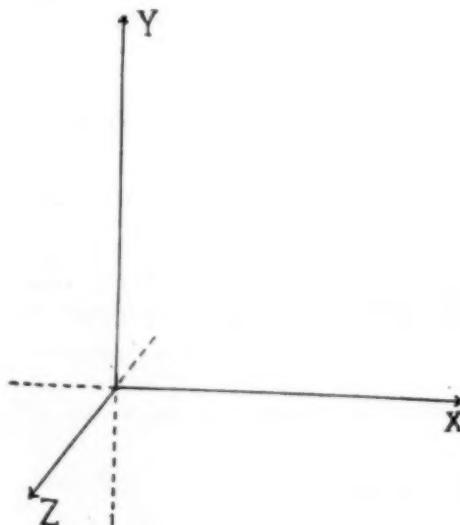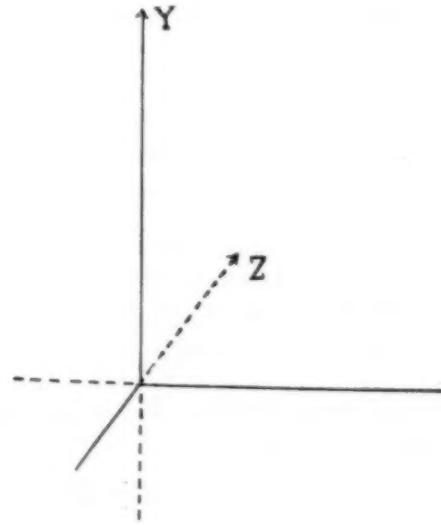
Figure 1.   Figure 2.

screw placed along the Z-axis will advance in the positive Z-direction when rotated from the positive X-direction toward the positive Y-direction. Similarly, proceeding in a cyclic way, we see that a rotation from Y toward Z gives a positive X-advance, and from Z toward X a positive Y-advance. The axes are said to form a *right-handed system*. Figure 2 represents a *left-handed system*.

143

These figures as drawn are not true perspectives, for it is only when we look directly down the Z-axis that the X- and Y-axes actually appear to be at right angles as we have shown them; nevertheless, such diagrams as these are frequently preferred because in some ways they are easier to use.

In plane analytic geometry the X-axis is traditionally taken to be directed to the right, the Y-axis upward, and rotation from the positive X-direction toward the positive Y-direction to be positive. Thus counter-clockwise rotation is positive. But this distinction depends on the fact that we are dealing with a one-sided plane; for if, in a bright light, we look at the same axes through the page we see that the positive direction is clockwise! For this reason, in *solid* analytic geometry angles are not considered algebraically; they are taken to be non-negative, usually between 0 and π radians, inclusive.

We have learned, in plane analytic geometry, to recognize the locus of such an equation as

$$y = x^2 \quad \text{or} \quad y = \sin x$$

as a *curve*, and have learned to appreciate the advantages of using functional notation to represent the equation of such a curve by

$$y = f(x) \quad \text{or} \quad y = g(x);$$

these formulas give y *explicitly* in terms of x. Sometimes y is given only *implicitly* as a function of x,

$$F(x,y) = 0,$$

either because more than one value of y might correspond to a single value of x or because the implicit equation has a simpler form; thus a circle with center at the origin and radius a can be represented by

$$x^2 + y^2 - a^2 = 0.$$

But the most flexible means of representing curves is by means of *parametric* equations,

$$x = \varphi(t), \quad y = \psi(t),$$

where the parameter t, used to determine coordinates (x,y) of points on the curve, might or might not have physical significance (such a elapsed time or arc length) in the generation of the curve; in any case, t does not appear as part of the graph itself. Thus the above circle can be represented by

$$x = a \cos t, \quad y = a \sin t.$$

What is the analogous situation in space? Consider the equation

(1) $$y = x^2 + z^2.$$

The intersection of the locus with the (X,Y)-plane, that is, the part of the locus for which $z = 0$, clearly is the parabola $y = x^2$, $z = 0$. Similarly, the intersection with the (Y,Z)-plane is the parabola $y = z^2$, $x = 0$. Indeed, if we picture the circle $x^2 + z^2 = r^2$, $y = 0$, in the (X,Z)-plane, and raise this circle a distance r^2 in the positive Y-direction, we obtain a circle on the locus, as shown in Figure 3.
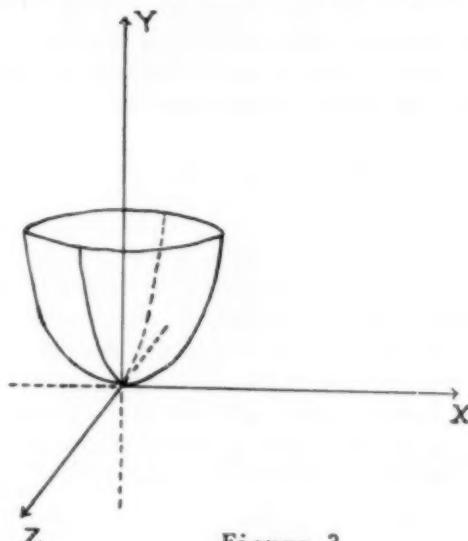

Figure 3.

As r increases beginning at zero, the *surface*, a circular paraboloid, is generated. As an exercise, you might determine the locus of

$$(2) \qquad y^2 = x^2 + z^2.$$

Thus a surface, like (1), for which there is exactly one value y corresponding to each (x,z) in the domain of definition, can be represented by an *explicit* equation of the form

$$(3) \qquad y = f(x,z).$$

More general is the *implicit* representation

$$(4) \qquad F(x,y,z) = 0$$

of a surface; for example, (2) can readily be put in this form, and

$$(5) \qquad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - a^2 = 0, \qquad a > 0,$$

is of this form. Since an immediate generalization to space of the plane distance formula is

$$(6) \qquad d = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{1}{2}},$$

you recognize (5) as an equation of a sphere with center (x_0, y_0, z_0) and radius a. But the most flexible way of representing surfaces in

space is by means of *parametric* equations

(7) $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$;

the equations

$$x = 3u + 3uv^2 - u^3,   y = 3v + 3u^2v - v^3,   z = 3u^2 - 3v^3,$$

are of the form (7).

Since two surfaces usually intersect in a curve, a curve in space may be represented as the locus of two simultaneous equations of the form (3) or (4), or of two simultaneous sets of equations of the form (7). Thus

$$x^2 + y^2 - 1 = 0,$$

and

$$x^2 + z^2 - 1 = 0,$$

represent two right circular cylinders in space; as simultaneous equations they represent the two elliptic curves of intersection of these two cylinders. Again, we have already discussed the parabola $y = x^2$, $z = 0$; it is the intersection of two surfaces: the parabolic cylinder $y = x^2$ (z not restricted), and the (X,Y)-plane $z = 0$ (x and y not restricted). But again the most flexible means of representing a curve in space is by means of parametric equations,

(8) $x = x(t)$, $y = y(t)$, $z = z(t)$.

For example, you can readily determine the interesting locus of

$$x = \cos t,   y = \sin t,   z = t/(2\pi).$$

In *differential geometry* we study properties of curves and surfaces by means of calculus. Accordingly differential geometry is, for the most part, concerned with properties of curves and surfaces in the immediate vicinity of a point; that is, differential geometry is, for the most part, a geometry *in the small*, and indeed we shall first discuss local properties of curves and surfaces. However, some properties of whole surfaces are implied by uniform local conditions, and differential geometry *in the large* is currently being exploited vigorously, as we shall indicate later. In *metric*, or *Euclidean*, *geometry*, as distinguished from topology or algebraic, projective, affine, or inversive geometry, we study properties of a geometric object relative to its size and shape. Thus *metric differential geometry* is the study by means of calculus of those properties of geometric objects, both in the small and in the large, which are invariant under rigid motions.

For two points P_1: (x_1, y_1, z_1), P_2: (x_2, y_2, z_2), let

$$\Delta x = x_2 - x_1,   \Delta y = y_2 - y_1,   \Delta z = z_2 - z_1,$$

let $|P_1 P_2| = r > 0$, and let the angles to the direction $P_1 P_2$ from the

positive X, Y, Z directions be denoted α, β, γ, respectively, as suggested in Figure 4. It is to be noted that α does not necessarily lie in a
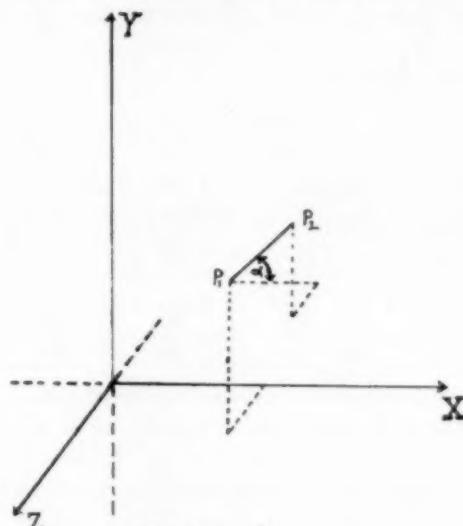


Figure 4.

plane parallel to the (X,Y)-plane. Then by the definition of the cosine function we have

$$(9) \qquad \frac{\Delta x}{r} = \cos\alpha, \qquad \frac{\Delta y}{r} = \cos\beta, \qquad \frac{\Delta z}{r} = \cos\gamma;$$

these are called the *direction-cosines* of the direction P_1P_2.

By the generalized Theorem of Pythagoras, we have

$$(10) \qquad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = r^2,$$

so that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Now let the above points P_1 and P_2 be points on a smooth curve C given by (8), with P_1 corresponding to t_1 and P_2 to t_2, and let $t_2 \rightarrow t_1$, with t_1 fixed; then $P_2 \rightarrow P_1$. If you understand the plane differential relation

$$ds^2 = dx^2 + dy^2,$$

where s denotes arc-length, then from (10) you will understand the space differential relation

$$(11) \qquad ds^2 = dx^2 + dy^2 + dz^2,$$

whence

$$s = \int_{t_1}^{t_2} \left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 \right]^{1/2} dt.$$

Just as in the plane we have

$$\frac{dy}{dx} = \tan \theta, \quad \text{and} \quad \frac{dx}{ds} = \cos \theta, \quad \frac{dy}{ds} = \sin \theta,$$

in space we have, from (9),

$$\left(\frac{dx}{ds}\right)_1 = \cos \alpha_1, \quad \left(\frac{dy}{ds}\right)_1 = \cos \beta_1, \quad \left(\frac{dz}{ds}\right)_1 = \cos \gamma_1;$$

these are the *direction-cosines of the tangent* to C at the point P_1. It is not necessary, however, to differentiate relative to the arc-length s in order to determine the direction of the tangent to C at P_1, for clearly

$$\frac{dx}{ds} : \frac{dy}{ds} : \frac{dz}{ds} = \frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt};$$

the latter are called *direction-components* of the tangent. You can satisfy yourself that the line tangent to C at P_1 is represented by

$$(12) \qquad x = x_1 + \left(\frac{dx}{dt}\right)_1 w, \quad y = y_1 + \left(\frac{dy}{dt}\right)_1 w, \quad z = z_1 + \left(\frac{dz}{dt}\right)_1 w,$$

where we have now chosen to represent the running parameter by w.

As P moves along C, the tangents to C generate an interesting surface, the *tangent surface* of C. Its equations are given by (12) when we replace x_1 by $x(t)$, $(dx/dt)_1$ by dx/dt, and so on; that is, when we replace the fixed value t_1 by the variable t. What are the two parameters of this surface?

A curve on a general surface S given by (7) might be determined by a curve

$$(13) \qquad\qquad u = u(t), \quad v = v(t)$$

in the (u,v)-domain of definition; for then

$$x = x[u(t), v(t)], \quad y = y[u(t), v(t)], \quad z = z[u(t), v(t)]$$

are functions of the single variable t. Since

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv, \quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv,$$

(11) becomes

$$(14) \qquad\qquad ds^2 = E du^2 + 2F du dv + G dv^2,$$

where

$$(15) \quad E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \quad F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \quad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

The values of the coefficients E, F, G are determined by (15) for any point P on S, and are the same for all curves through P on S; but the relation of du to dv depends on the particular curve (13). Each

curve on S through P has a tangent line at P, and the totality of these tangent lines constitutes a plane, the *tangent plane* Q to S at P.

Continuing the analysis by means of differential calculus, we learn many additional interesting facts concerning the local behavior of curves and surfaces. Thus relative to the behavior of a twisted curve C in the neighborhood of one of its ordinary points P we learn about such appropriately named entities as osculating plane, osculating circle or circle of curvature, torsion, principal normal, and binormal. Roughly, the *osculating plane* is the plane in which the part of C near P most nearly lies (has contact of highest order), the *circle of curvature* is the circle (in the osculating plane) which most closely fits C at P, the *torsion* is a measure of the rate at which C is twisting out of its osculating plane at P, the *principal normal* is the line in the osculating plane perpendicular to the tangent to C at P, and the *binormal* is the line through P perpendicular to the tangent and also to the principal normal.

The behavior of C near P is best exhibited when its equations are given relative to the tangent, the principal normal, and the binormal *as coordinate axes*. If you can imagine three mutually perpendicular axes moving along C, one of them directed always along the tangent to C, and twisting so that another lies always in the osculating plane, then you can visualize the fundamental *moving trihedral* of C.

We shall use the above notion of the circle of curvature of a space curve C at one of its points P in discussing only a few of the many interesting properties of surfaces. We have pointed out that at a point P of a smooth surface S there is a unique tangent plane Q. Accordingly there is a unique line N normal (perpendicular) to S at P. You can visualize the pencil of planes Q_θ, $0 \leq \theta < \pi$, with N as axis, and can see that each of these planes Q_θ cuts S in a corresponding curve C_θ. Now each of the curves C_θ has a circle of curvature, called the *circle of normal curvature* of S in the direction θ.

To fix the picture, think of two points on the surface (skin) of your clinched fist: a point P_1 at the peak of one of your knuckles and a point P_2 on the floor of the valley between two knuckles. The surface is quite different in the neighborhoods of these two points—near P_1 the surface lies entirely on one side of its tangent plane, while in any neighborhood of P_2 the surface is cut by its tangent plane! In the first case the center of the circle of normal curvature lies within your fist for all directions θ, while in the second case two *asymptotic curves* through P on S separate the directions for which the center of normal curvature lies within your fist from those for which it lies in space outside your fist. We might give an algebraic sign to the radius of the circle of normal curvature by saying that it is positive if the center lies on one side of the surface (say within your fist), and negative it lies on the other side (outside your fist). With such a convention as to algebraic sign, the radius of the circle of normal curvature in the

direction θ is called the *radius of normal curvature* of S at P in the direction θ, and its reciprocal is called the *normal curvature* of S at P in this direction. Thus, for instance, in an asymptotic direction the normal curvature has the value zero.

Now an interesting fact emerges: there is one direction θ_1 for which the normal curvature of S at P has a maximum $1/r_1$, and another direction θ_2 for which it has a minimum $1/r_2$, and *these directions are at right angles to each other*. The values $1/r_1$ and $1/r_2$ are called the *principal normal curvatures*, and θ_1 and θ_2 the *principal directions*, on S at P. In the exceptional case that $1/r_1 = 1/r_2$, all directions on S at P are considered to be principal directions, and P is said to be an *umbilical point* of S. All points of S are umbilical if and only if S is a plane or spherical surface.

For each of the above points P_1 and P_2 on your fist, the principal directions are more or less 1) in the direction of your fingers, and 2) in the direction across your knuckles.

Very important functions in the theory of surfaces, because of their geometrical significance and analytical expressions, are the product and the sum of the principal curvatures,

$$K = \frac{1}{r_1 r_2}, \quad K_{\blacksquare} = \frac{1}{r_1} + \frac{1}{r_2},$$

which are called the *total*, or *Gaussian*, *curvature*, and the *mean curvature*, respectively. If $K > 0$ at a point, then $1/r_1$ and $1/r_2$ are both of the same sign there, and the part of the surface near the point lies on only one side of the tangent plane, as at the above point P_1 on your fist. On the other hand, the part of your fist near the point P_2 illustrates the behavior of a surface in the neighborhood of a point where $K < 0$; such a point is called a *saddle-point*, for an obvious reason.

Another surface well illustrating points of both the above sorts is the torus (doughnut), which you can approximate roughly by joining the tips of your thumb and forefinger. The outer points, where $K > 0$, are separated from the inner points, where $K < 0$, by two circles on which $K = 0$.

Surfaces, like ellipsoids, for which $K > 0$ at each point, are called *surfaces of positive curvature*. If $K < 0$ at each of its points, a surface is said to be a *surface of negative curvature*, or a *saddle surface;* a good illustration is the hyperbolic paraboloid,

$$y = x^2 - z^2.$$

A surface on which $K \equiv 0$ is called a *developable surface*, since in the small it can, like the right circular cylinder, be rolled out, or developed, on a plane surface.

Surfaces of constant *mean* curvature $K_{\blacksquare}$ are of considerable physical importance, particularly those for which $K_{\blacksquare} \equiv 0$. These last surfaces $(K_{\blacksquare} \equiv 0)$ are called *minimal surfaces*, because they are the surfaces for which the first variation (in the terminology of the Calculus of Variations)

of the area integral vanishes. The smooth surface of least area spanning
a given contour in space is necessarily a minimal surface. Clearly at
each point of a minimal surface we have $K \leq 0$, so that minimal surfaces
are special surfaces of non-positive Gaussian curvature. The theory of
minimal surfaces furnishes a beautiful generalization to space of much
of the fundamental theory of analytic functions of a complex variable.

The problem, to which we have alluded in the preceding paragraph, of
determining a minimal surface spanning a given space contour, is called
the *problem of Plateau* after the Dutch physicist who solved the problem
empirically for certain contours by means of soap-film experiments.
Though concerned with the differential condition $K_{\blacksquare} \equiv 0$, this is *a problem
in the large*.

Differential geometry in the large is concerned with relations between
differential, or local, properties of a geometric object and properties
of the object as a whole. It is not surprising that the following illus-
tration involves *integration;* in it we shall be concerned with *length*,
given by

$$l = \int_{t_1}^{t_2} \left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 \right]^{\frac{1}{2}} dt,$$

and *area*, given by

$$a = \iint_D (EG - F^2)^{\frac{1}{2}} du\, dv.$$

For a given contour length l in the plane, it is intuitively clear,
though not particularly easy to prove, that the circle has the greatest
area. Since for the circle we have

$$a = \pi r^2 = \frac{1}{4\pi}(2\pi r)^2 = \frac{1}{4\pi} l^2,$$

for general *Jordan regions*, or regions bounded by simple closed curves,
in the plane we have

$$(16) \qquad\qquad a \leq \frac{1}{4\pi} l^2;$$

this is called the *isoperimetric inequality*. Now (16) does not hold for
all regions bounded by curves on surfaces: a tiny circle on an arbi-
trarily large sphere ($K \equiv$ const. > 0) bounds two regions, one of which has
arbitrarily large area. However, it can be shown that if a surface
"bends both ways" at each point, then (16) holds. Indeed, *a necessary
and sufficient condition that* (16) *hold for each Jordan region on a
surface S is that S be a surface of non-positive Gaussian curvature*.

Two surfaces S_1 and S_2 having the same (u,v)-domain of definition,
with

$$ds_1{}^2 = E_1\, du^2 + 2F_1\, du\, dv + G_1\, dv^2,$$

and

$$ds_2{}^2 = E_2\, du^2 + 2F_2\, du\, dv + G_2\, dv^2,$$

are said to be *applicable* provided $E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$. Roughly, S_1 and S_2 are applicable provided each small portion of S_1 can be bent without stretching or tearing in such a way as to be made to coincide with the corresponding small portion of S_2; accordingly, any property of a surface S which depends only on E, F, and G is independent of the bending of the surface. Such a property is called an *intrinsic* or *absolute* property, and the geometry of these properties is called *intrinsic* or *absolute geometry*. The length of a curve is intrinsic, while its curvature certainly is not. The area of a surface is intrinsic, though its mean curvature is not. Surprisingly, *the Gaussian curvature of a surface is intrinsic;* thus, as we noted before, any surface with $K \equiv 0$ is developable.

Since absolute geometry is concerned with the surface itself rather than with the containing space, in this geometry the origin (7) of the functions E, F, and G is not important. Instead, we might start with any quadratic form, often restricted to being positive definite,

$$ds^2 = g_{11}\,du^2 + 2g_{12}\,du\,dv + g_{22}\,dv^2,$$

where the g_{ij} are functions of u and v, or more generally with

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\,du^i du^j, \qquad g_{ij} = g_{ji},$$

and develop the corresponding geometry, called *Riemannian geometry*. We use the methods of *absolute differential calculus*, or *tensor calculus*, in the development of this geometry. These are the tools that are used in the *general theory of relativity*.

The theory of differential geometry is often given as a separate course in advanced or graduate mathematics. It can also be studied as a principal application in a course in vector or tensor analysis, or can be learned quite well through independent reading. In any case, the teacher of high school geometry should find it valuable supplementary material; and most graduate mathematicians and many physicists find it desirable to learn the fundamentals, and specialists in several fields to acquire a mastery, of the subject early in their careers.

National Bureau of Standards and
University of California, Los Angeles

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the
MATHEMATICS MAGAZINE. lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting.
aspects further illuminated, comments on published papers in all departments
are invited.

Communications intended for this department should be sent in duplicate to
H. V. Craig, Department of Applied Mathematics, University of Texas, Austin
12, Texas.

Introduction to Complex Variables and Applications. By Ruel V.
Churchill, McGraw-Hill Book Co., New York, 1948, vi + 216 pages, $3.50.

The fallacious notion that there is one type of mathematics for
physical chemists, another type for biologists, another type for engi-
neers, etc., has perhaps gained some acceptance from the number of
published titles suggesting such diversities. In many instances the
authors of such books have not been mathematicians, and these books at
best have given an incomplete picture of mathematics.

Lately, counteracting tendencies have been noticed. There is a
growing tendency to centralize, within a university, all of the elemen-
tary statistical teaching.[1] Many mathematicians employed by industrial
concerns (who might be supposed to be the leaders in a cry for math-
ematics for special groups) have deplored the departure of mathematical
courses from the content of mathematics.[2]

Prof. Churchill's book *Introduction to Complex Variables and
Applications* is admittedly intended for students in physics, theoretical
engineering and applied mathematics. In many respects, it is a companion
volume to two previous volumes by the same author and the new book no
doubt will have wide appeal among those who have profited from use of
the earlier books.

In this book, examples from engineering and physics have been kept
at an elementary level and have been used to illustrate the mathematical
concepts. Emphasis has been placed on the mathematical content. The
author's unwillingness to jump at once into the possible applications
to the solving of text-book type problems is indicated by a paragraph
from page 74:

"The reader may pass directly to the chapters on conformal
mapping and applications at this time, if he wishes. It would
seem natural to present that chapter next, since we have just
completed a study of mappings by elementary functions. However,
we have not yet established the continuity of the second-order

derivatives of the real and imaginary components, $U(x,y)$ and $V(x,y)$, of an analytic function. If we take up the subject of the conformal mapping at this time, we shall have to assume that continuity; to establish it we need the use of some of the theory of integrals of analytic functions, which is presented *below.*"

The "below" referred to consists of sixty pages of mathematical textual matter. It is to be hoped that the engineering reader does not pass directly to the chapters on conformal mappings and applications for then he would doubtless encounter difficulty when trying to solve his non-textbook type problems presented by engineering. Many such individuals in later life might be tempted to say that mathematics had been of no use to them in their technical careers. But who are they to make such statements, seeing that they have had no mathematics?

Professor Churchill's book may therefore be considered as a valuable addition to the mathematical textbooks for giving training and instruction in mathematics to future physicists and engineers. Since it is intended as a one-semester course outline, it is inadequate for future mathematicians, and, of course, the author makes no claim to any such sufficiency.

Algebraic concepts supplemented by geometrical concepts are used to present the notion of a complex number. The classical theory of functions of a complex variable is presented first from the viewpoint of line integrals generally associated with the name of Cauchy and later supplemented with a power series treatment closely associated with the name of Weierstrass. The notion of Riemann surfaces, representing perhaps a third approach, is the subject of the last (and short) chapter.

The reviewer noticed some misprints which in his opinion do not impair the usefulness of the book.

Robert E. Greenwood

[1] E. L. Lehmann, *Amer. Math. Monthly* v. 56 (1949) p. 430.

[2] This attitude was exemplified in an address made by Dr. M. M. Slotnick of the Humble Oil and Refining Co. to the Texas Section of the Mathematical Association of America meeting in Denton, Texas in April 1949.

BOOKS RECEIVED

The Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications. By Norbert Wiener. New York, John Wiley and Sons, Inc., 1949. IX + 163 pages. $4.00.

Partial Differential Equations in Physics. By Arnold Sommerfeld. New York, Academic Press Inc. Publishers, 1949. XI + 335 pages. $5.80.

Topology of Manifolds (American Mathematical Society Colloquium

Publications, vol. XXXII). By R. L. Wilder. New York, American Mathematical Society, 1949. IX + 404 pages. $7.00.

The Geometry of the Zeros of a Polynomial in a Complex Variable (Mathematical Surveys, Number III). By Morris Marden. New York, American Mathematical Society, 1949. IX + 183 pages, $5.00.

Giant Brains. By Edmund C. Berkeley. New York, John Wiley and Sons Inc., 1949. XVI + 270 pages. $4.00.

A Short Course in Differential Equations. By Earle D. Rainville. New York, The Macmillan Co., 1949. IX + 210 pages. $3.00.

First Year College Mathematics with Applications. By Paul H. Daus and William M. Whyburn. New York, The Macmillan Co., 1949. XIII + 495 pages.

First Year Mathematics for Colleges. By Paul R. Rider. New York, Macmillan Co., 1949. XV + 714 pages. $5.00.

Analytic Geometry. By W. A. Wilson and J. I. Tracey. Boston, D. C. Heath and Co., 1949. X + 318 pages. $2.75.

Analytic Geometry. By John J. Corliss, Irwin K. Feinstein, and Howard S. Levin. New York, Harper and Brothers Publishers, 1949. XIV + 370 pages.

Analytic Geometry. By Charles H. Sisam. New York, Henry Holt and Co., 1949. XVI + 304 pages. $2.40.

Analytic Geometry. By Robin Robinson. New York, McGraw-Hill Book Co., 1949. IX + 147 pages. $2.25.

Mathematics of Finance. By Clifford Bell and L. J. Adams. New York, Henry Holt and Co., 1949. VII + 366 pages. $2.75.

Mathematics for Finance and Accounting. By J. B. Coleman and William O. Rogers. New York, Pitman Publishing Corporation, 1949. X + 310 pages.

College Algebra. By Joseph B. Rosenbach and Edwin A. Whitman. Boston, Ginn and Co., 1949. X + 523 + XLII pages.

Commercial Algebra. By Clifford Bell and L. J. Adams. New York, Henry Holt and Co., 1949. VII + 304 pages. $2.75.

PROBLEMS AND QUESTIONS
Edited by
C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known text-books should not be submitted.

In order to facilitate their consideration, solutions should be submitted on separate, signed sheets within three months after publication of the problems.

All manuscripts should be typewritten on 8½″ by 11″ paper, double-spaced and with margins at least one inch wide. Figures should be drawn in india ink and in exact size for reproduction.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 27, Calif.*

PROPOSALS

49. *Proposed by F. C. Gentry, Arizona State College at Tempe.*

The product of the radii of either group of adjoint circles of a triangle is equal to the cube of the circumradius.

50. *Proposed by Alan Wayne, Flushing, N. Y.*

In the addition *THREE + EIGHT + NINE = TWENTY* no two different letters represent the same digit. Find the value of each letter and show that with $R > G$ the solution is unique.

51. *Proposed by E. P. Starke, Rutgers University.*

Find the smallest number (>1) of distinct, odd positive integers such that the sum of their reciprocals is unity. Determine such a set.

52. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

If a plane cuts the surface and volume of a tetrahedron into two equivalent parts, this plane passes through the center of the inscribed sphere of the tetrahedron.

53. *Proposed by F. L. Miksa, Aurora, Ill.*

Find the shortest perimeter common to four different primitive Pythagorean triangles.

54. *Proposed by Howard Eves, Oregon State College.*

If one side of a given complete quadrilateral is parallel to the Euler line of the triangle formed by the remaining three sides, the same is true for every side of the quadrilateral.

55. *Proposed by R. W. Winger, Los Angeles City College.*

(a) A cylindrical bucket of radius r and height h is $1/k$ full of liquid. Find the maximum speed at which the bucket may be whirled about its vertical axis of symmetry without losing any of the liquid. Neglect surface tension. (b) What is the maximum value of k for which no part of the bottom of the bucket will be dry at the maximum speed?

SOLUTIONS

Correction

5. [Sept. 1947] The forty-first value of M given in THIS MAGAZINE, **23**, 105, (Nov. 1949) should be 96895 not 96985.

The Limaçon as a Locus

14. [Jan. 1948] *Proposed by A. K. Waltz, Clarkson College, Potsdam, N.Y.*

The tangent of an angle, whose sides slide on two given circles, has constant magnitude. Determine and discuss the locus of its vertex.

Solution by J. H. Butchart, Arizona State College at Flagstaff.

This locus is discussed in my note, Some Properties of the Limaçon and Cardioid, *American Mathematical Monthly*, **52**, 384-7, (1945). There the following proof may be found, together with a figure and a discussion of special cases.

We might begin by assuming the angle whose sides slide on the circles in an initial position, but it seems more convenient to use an indirect approach. Let C be a point on the circle of similitude (S). Let Q be any point on the circle through $O'CO$, where (O) and (O') are the given circles, and let TP, $T'P'$ be tangents to (O), (O') parallel respectively to OQ, $O'Q$ and meeting CQ in P, P' such that these points are both outside in the same direction. Then since angles CPT and $CP'T'$ are constant, as they are equal to CQO and CQO' respectively, the segments QP and QP' are constant. When Q is in the special position diametrically opposite C, then P and P' coincide, since then $CO{:}CT{:}{:}CO'{:}CT'$. Thus P and P' are identical in all positions and P describes a limaçon with with double point at C.

That this discussion covers all possibilities may be seen by noting that the two tangents to (O) parallel to OQ meet the two tangents to (O') parallel to $O'Q$ in four points two of which lie on CQ and the other two on $C'Q$, where C' is the other point common to (S) and (OQO'). The two limaçons whose double points are C and C' are evidently not in general congruent. If we were asking for the locus of all points from which tangents to the respective fixed circles can be drawn intersecting at a given angle, we should have to include two more limaçons symmetrical to these with respect to the line of centers.

Also solved by K. L. Cappel, San Francisco, California and the proposer.

Other references pertaining to this problem are: *Mathesis*, **1**, 12, 43, (1881); *L'Intermédiaire des Mathématiciens*, **9**, 117, 286, (1902); *American Mathematical Monthly*, **43**, 579, (1936); **50**, 123, (1943); Williamson, *Differential Calculus*, page 372; Yates, *Curves and Their Properties*, J. W. Edwards (1947), page 149.

Iterated Summation of Consecutive Integer Squares

22. [Sept. 1948] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Let x and n be any two positive integers and let $\Sigma^n x^2$ stand for the nth iterated summation of all the squares from 1 to x^2 inclusive. For instance $\Sigma 4^2 = 30$, $\Sigma^2 4^2 = 50$, (that is, the sum of the sums of all the squares from 1 to 16 inclusive), and $\Sigma^5 4^2 = 156$ (that is, the sum of the sums of the sums of the sums of all the squares from 1 to 16 inclusive). Prove that in general

$$\Sigma^n x^2 = x(x+1)(x+2)(x+3) \cdots (x+n)(2x+n)/(n+2)!$$

Solution by E. P. Starke, Rutgers University. We have

$$x^2 = 1 + 3 + \cdots + (2x-1) = \Sigma(2x-1) = \Sigma\left\{ \binom{x-1}{1} + \binom{x}{1} \right\}$$

By use of the familiar formula (easily proved)

$$\Sigma\binom{s}{r} = \binom{s+1}{r+1} ,$$

we obtain

$$x^2 = \binom{x}{2} + \binom{x+1}{2} ,$$

whence

$$\Sigma^1 x^2 = \binom{x+1}{3} + \binom{x+2}{3} ,$$

and with continued repetition of this process we have finally,

$$\Sigma^n x^2 = \binom{x+n}{n+2} + \binom{x+n+1}{n+2} = \frac{(x+n)!(2x+n)}{(n+2)!(x-1)!} ,$$

which is the proposer's result.

Solutions of this problem by other methods appeared in THIS MAGAZINE, **22**, 105, (Nov. 1948) and **22**, 163, (Jan. 1949).

Obscuration of the Celestial Hemisphere

33. [Jan. 1949] *Proposed by F. M. Steadman, Los Angeles, California.*

Let R denote the proportion of the upper celestial hemisphere obscured at a point P by a horizontal circular area of radius a whose center is vertically above P at a distance x. Express x in terms of a and R. In particular, find x in centimeters when the diameter of the circular area is one meter for $R = 1/2$, $1/4$, $1/8$.

The usual measure of solid angles is entirely analogous to the radian measure of plane angles. Is there a possibly useful measure of solid angles which is analogous to degree measure of plane angles and applicable to measurement of lens stops in photography?

I. *Solution by Hugh Hamilton, Los Angeles City College.* Consider the sphere about P of radius $\sqrt{a^2 + x^2}$. By the well-known formula for the area of a spherical zone, we have $R = (\sqrt{a^2 + x^2} - x)/\sqrt{a^2 + x^2}$, and from this we obtain $x = a(1-R)/\sqrt{R(2-R)}$. If the diameter of the

circular area is one meter, then $a = 50$ cm. and the above formula yields $x = 28.9$ cm. for $R = 1/2$; $x = 56.7$ cm. for $R = 1/4$; and $x = 90.4$ cm. for $R = 1/8$.

II. *Solution by Erich Marchand, Eastman Kodak Co., Rochester, N. Y.* The problem is essentially that of finding the area cut out of a sphere by a right circular cone with vertex at the center. This can be done by high-school geometry or, if cylindrical coördinates are used, the area is given by $S = \int_A \int \rho \, d\rho \, d\theta / \cos \Upsilon$, where A is the projection of the area S perpendicular to the axis of the cone, and $\cos \Upsilon = \sqrt{1 - \rho^2 / r^2}$, r being the radius of the sphere.

Evaluating the integral, we have

$$S = \int_0^{2\pi} \int_0^{\rho_0} r \rho \, d\rho \, d\theta / \sqrt{r^2 - \rho^2}, \quad \text{where} \quad \rho_0 = ar / \sqrt{a^2 + x^2}.$$

This leads to $S = 2\pi r^2 (1 - x/\sqrt{a^2 + x^2})$ and $R = S/2\pi r^2 = 1 - x/\sqrt{a^2 + x^2}$. From this $x = a(1 - R)/\sqrt{2R - R^2}$. For $a = 50$ cm. and $R = 1/2$, $1/4$, and $1/8$ we have $x = 50/\sqrt{3}$ or 28.9 cm., $150/\sqrt{7}$ or 56.7 cm., and $350/\sqrt{15}$ or 90.4 cm., respectively.

It is entirely possible to define arbitrarily a unit of solid angle in terms of degree measure of plane angles and in a sense this definition of a unit of solid angle is analogous to the usual measure of solid angle which is related to the radian measure of plane angles. Thus a unit of solid angle might be arbitrarily defined as the solid angle at the apex of a circular cone having a plane angle of one degree. This might conceivably be referred to as a "Sterdegree." The analogy of such a unit with the steradian, the usual unit in terms of which solid angle is expressed, is not complete since a steradian is not defined as a solid angle at the apex of a circular cone having a plane angle of one radian.

The method at present employed for designating the size of a lens stop involves the use of the f number. The illumination incident on the focal plane of the camera is directly proportional to $1/f^2$, the reciprocal of the square of the f number. If lens stops were marked in terms of the solid angle subtended by the aperture at a point on the focal plane of the camera the illumination would not be directly proportional to the solid angle [see, Hardy and Perrin, *The Principles of Optics*, McGraw-Hill (1932), page 415] regardless of whether that solid angle were expressed in steradians, millisteradians, or sterdegrees. Since this is true, it is doubtful that any unit of solid angle would be as useful as the f number for the measurement of lens stops in photography.

Area Commanded by a Gun

36. [March 1949] *Proposed by J. S. Miller, Michigan College of Mining and Technology.*

A gun can put a projectile to a height R/n, R being the radius of the Earth. Assuming the variation in gravitational force with altitude, find the area commanded by the gun.

Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C. According to Newton's law of gravitation $g = g_0 (R/r)^2$, where g_0 is the acceleration due to gravity at sea level and r is the distance from the center of the Earth. If u is the striking velocity of a body falling from a point P in space, its kinetic energy per unit mass at impact with the Earth is $u^2/2$. This must be equal to the potential energy of the body at the point P. That is,

$$u^2/2 = \int_R^{R+R/n} g\,dr = \mu \int_R^{R+R/n} r^{-2}\,dr = \mu/R(n+1), \quad \text{or}$$

$$u^2 = 2\mu/R(n+1), \quad \text{where } \mu = g_0 R^2. \tag{1}$$

Thus (1) gives the initial velocity for a projectile fired vertically to reach the height R/n assuming the variation of gravity with the distance from the center of the earth but neglecting air resistance.
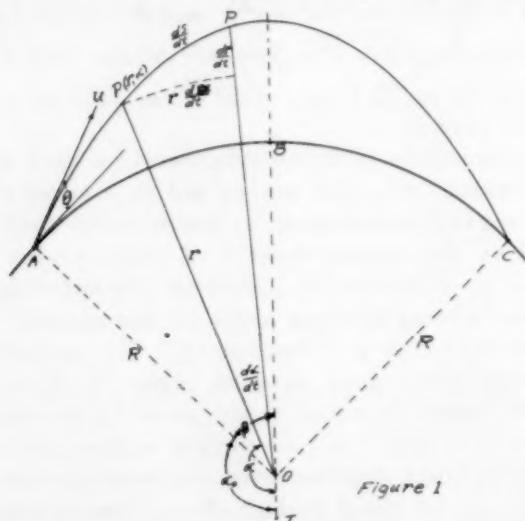
Figure 1

We now assume the polar coördinate system as shown in Figure 1. θ is the angle of elevation of the gun at the point A. The origin of coördinates is the center, O, of the Earth and the angle α is measured clockwise from the direction OT as shown. The polar coördinates of a point P on the trajectory then are r, α as shown.

We may use Kepler's second law and write from Figure 1, $r^2 d\alpha/dt = k$, a constant. When P is at A, $r\,d\alpha/dt = (ds/dt)\cos\theta = u\cos$ and $r = R$, hence

$$r^2 d\alpha/dt = Ru\cos\theta = k. \tag{2}$$

In the motion of the shell along the trajectory, we have, from the integral used to obtain (1), (with the sign changed), the integral of the velocity from A to P. That is,

$$v^2\Big]_u^v = -2\mu\int_R^r r^{-2}\,dr, \quad \text{or}$$

$$v^2 = u^2 + 2\mu/r - 2\mu/R = q + 2\mu/r, \quad \text{where } q = u^2 - 2\mu/R. \tag{3}$$

From Figure 1 we have

$$v^2 = (ds/dt)^2 = (dr/dt)^2 + r^2(d\alpha/dt)^2. \tag{4}$$

But $dr/dt = (dr/d\alpha)(d\alpha/dt)$ and from (2), $d\alpha/dt = k/r^2$, so that $dr/dt = (k/r^2)(dr/d\alpha)$ and we may write (4) as

$$v^2 = (ds/dt)^2 = (k^2/r^4)(dr/d\alpha)^2 + k^2/r^2. \tag{5}$$

Now combining (3) and (5) and solving for $dr/d\alpha$ we obtain the polar differential equation of the trajectory.

$$d\alpha = k\, dr/r^2\sqrt{q + 2\mu/r - k^2/r^2}. \tag{6}$$

Equation (6) may be manipulated into the form

$$d\alpha = -d\left[\frac{k/r - \mu/k}{\sqrt{q + \mu^2/k^2}}\right] \Bigg/ \sqrt{1 - \left[\frac{k/r - \mu/k}{\sqrt{q + \mu^2/k^2}}\right]^2},$$

which integrates to give

$$\alpha - \alpha_1 = \arccos (k/r - \mu/k)/\sqrt{q + \mu^2/k^2}, \quad \text{or}$$

$$r = p/[1 + e \cos(\alpha - \alpha_1)], \quad \text{where } p = k^2/\mu, \quad e = \sqrt{1 + qk^2/\mu^2}. \tag{7}$$

This is clearly the polar equation of a conic section with one focus at the center, O, of the Earth. If $\alpha_1 = 0$, then the polar axis OT of the system is the line joining the perihelion, T, to the center of the earth, and equation (7) is finally

$$r = p/(1 + e \cos \alpha). \tag{8}$$

Equation (8) represents an ellipse, a parabola, a hyperbola according as $e \lesseqgtr 1$ and a circle if $e = 0$.

We have

$$e = \sqrt{1 + qk^2/\mu^2} = \sqrt{1 + R^2u^2\cos^2\theta(u^2 - 2\mu/R)/\mu^2}. \tag{9}$$

From (9) it is seen that $e \lesseqgtr 1$ according as $u^2 \lesseqgtr 2\mu/R$. Now $\sqrt{2\mu/R}$ is the velocity of escape (about 7 miles/second) and for the practical solution of our problem we assume that $u^2 < 2\mu/R$. The case $e = 0$ leads to the condition $Ru^2/\mu = 1 \pm \sqrt{1 - \sec^2\theta}$, whence $\cos\theta = \pm 1$, $\theta = 0$ or π. That is, the shell must be fired horizontally (in the tangent to the surface at the point A) with the initial squared velocity $u^2 = \mu/R$ which is one-half the squared velocity of escape. We have with this condition and equation (1) that $2/(n+1) = 1$, which is true only if $n = 1$. We also reject this case as being untenable for guns so the trajectory is then, for our problem, an ellipse for all $n > 1$.

From an inspection of Figure 1 and equation (8), the arc ABC, or the range, is given by $2R\beta$. But β is the supplement of α_0 and from (8)

$$\alpha_0 = \arccos (p - R)/eR. \tag{10}$$

Since $p = k^2/\mu = (u^2/g_0)\cos^2\theta$, and $e = \sqrt{1 + u^2(u^2 - 2g_0 R)\cos^2\theta/g_0^2 R^2}$, we may write (10), using the value of u^2 from (1), in the form

$$\alpha_0 = \arccos(\cos 2\theta - n)/\sqrt{n^2 - 2n\cos 2\theta + 1}. \tag{11}$$

From (11),

$$d\alpha_0/d\theta = 2(1 - n\cos 2\theta)/(n^2 - 2n\cos 2\theta + 1). \tag{12}$$

The denominator of (12) cannot be zero for $n > 1$, hence from the numerator we have $\cos 2\theta = 1/n$. Returning this value to (11) gives $\alpha_0 = \arccos(-\sqrt{1 - 1/n^2})$. Therefore the maximum range is

$$S = 2R[\pi - \arccos(-\sqrt{1 - 1/n^2})], \quad n > 1.$$

It follows that the area commanded by the gun is the area of the spherical zone generated about the gun by the maximum range. That is, $A = 2\pi R^2(1 - \cos 2\beta) = 2\pi R^2[1 - \cos 2\arccos(-\sqrt{1 - 1/n^2})] = 4\pi R^2/n^2$.

Note: See C. Cranz and K. Becker, *Handbook of Ballistics*, Volume I.

MATHEMATICAL MISCELLANY

Edited by

Charles K. Robbins, Purdue University

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other ''matters mathematical'' will be welcome. Address: CHARLES K. ROBBINS. Department of Mathematics, Purdue University, Lafayette, Indiana.

It is a curious fact that toward the end of the century some of the leading mathematicians expressed the feeling that the field of mathematics was somehow exhausted. The laborious efforts of Euler, Lagrange, D'Alembert, and others had already led to the most important theorems; the great standard texts had placed them, or would soon place them, in their proper setting; the few mathematicians of the next generation would only find minor problems to solve. ''Ne vous semble-t-il pas que la haute géometrie va un peu à décadence?'' wrote Lagrange to D'Alembert in 1772. ''Elle n'a d'autre soutien que vous et M. Euler.'' (''Does it not seem to you that the sublime geometry tends to become a little decadent? She has no other support than you and Mr. Euler.'' ''Geometry'' in Eighteenth Century French is used for mathematics in general. Lagrange even discontinued working in mathematics for a while.

D'Alembert had little hope to give. Arago, in his "Eloge of Laplace" (1842) later expressed a sentiment which may help us to understand this feeling:

"Five geometers — Clairaut, Euler, D'Alembert, Lagrange and Laplace — shared among them the world of which Newton had revealed the existence. They explored it in all directions, penetrated into regions believed inaccessible, pointed out countless phenomena in those regions which observation had not yet detected, and finally — and herein lies their imperishable glory — they brought within the domain of a single principle, a unique law, all that is most subtle and mysterious in the motions of the celestial bodies. Geometry also had the boldness to dispose of the future; when the centuries unroll themselves they will scrupulously ratify the decisions of science."

Arago's oratory points to the main source of this "fin de siècle" pessimism, which consisted of the tendency to identify the progress of mathematics too much with that of mechanics and astronomy. From the times of ancient Babylon until those of Euler and Laplace astronomy had guided and inspired the most sublime discoveries in mathematics; now this development seemed to have reached its climax. However, a new generation, inspired by the new perspectives opened by the French Revolution and the flowering of the natural sciences, was to show how unfounded this pessimism was. This great new impulse came only in part from France; it also came, as often in the history of civilization, from the periphery of the political and economical centers, in this case from Gauss in Göttingen.

A Concise History of Mathematics Vol. II, Dirk J. Struik, Massachusetts Institute of Technology.

A Prime Representing Function

Consider the number $A = .203005000700011...$, where the n digits extending from the $\frac{n(n-1)}{2}$ place to the $\frac{n(n+1)}{2}$ place form the decimal representation of the nth prime P_n. The possibility of this number, i.e. the non-overlapping of the primes, is assured by Bertrand's postulate $P_{n+1} < 2P_n$. Then we have the formula

$$P_n = A \cdot 10^{\frac{n(n+1)}{2}} - 10^n A \cdot 10^{\frac{n(n-1)}{2}}.$$

Clearly the right hand member is just the integer represented by the n digits from the $\frac{n(n-1)}{2}$ place to the $\frac{n(n+1)}{2}$ place.

In this expression the 10 may be replaced by 3 or even by 2 with a slight modification in the expression. It is also easy to prove A irrational. While admittedly rather trivial, this result is of some

interest in connection with the statement in Hardy and Wright that a ''general formula'' for P_n is ''quite unreasonable.''

University of North Carolina   Leo Moser

Student in mathematics class: ''Professor, I understand this course perfectly; its just the mathematics that bothers me.''

''The advent of non-Euclidean geometry is, I have said, one of the gravest events in the history of thought. It has been tragic as well. The two facts are connected. Thirty years ago, I visited a locally eminent professor of mathematics in an excellent middle-west college of the sectarian variety . I was astonished to find him in a sad mental state, worried, distracted, agitated, tremulous, unable to sleep or rest, thinking always about the same thing, and no longer able to do so coherently. What was the trouble? For many years he had been teaching geometry, — Euclidean geometry, — and his teaching had been done in the spirit of a venerable philosophy. Like almost all the educated men of his time and like millions of others in the preceding centuries, he had been bred in the belief that the geometry he was teaching was far more than a body of logical compatibilities; it was not only true internally, — logically sound, that is, — but it was true externally — an exact account of space, the space of the sky and the stars; its axioms were not mere assumptions, — not mere *ifs*, — they were truths, ''self-evident'' truths, and, like the propositions implied by them, they were not only valid but were known to be valid, and valid eternally; in a word, the geometry of Euclid was body of *absolute knowledge* of the nature of space, — the space of the outer world, — other space there was none. That was a comforting belief, a congenial philosophy, held as a precious support of religion and life; for, though there are many things unknown and some perhaps unknowable, yet *something*, you see, was known; there was thus a limit to rational scepticism; our human longing for certitude had at least one great gratification — the validity of Euclidean geometry as a description of Space was indubitable. Such was the philosophy in which my dear old friend had been bred, and with unquestioning confidence, he had devoted long years to the breeding of others in it. At length, he heard of non-Euclidean geometries, in which his cherished certitudes were denied -- denied, he knew, by *great* mathematicians, by men of creative genius of the highest order; he could not accept, he could not reject, he could not reconcile; the foundations of rational life seemed utterly destroyed; he pondered and pondered but the great new meaning he was too old to grasp, and his mind perished in the attempt,— killed by the advancement of science, — slain by a revolution of thought.'' — C. J. Keyser, Mathematical Philosophy, N. Y. 1922, 363-5.

THE CHRONOLOGY OF PI

Herman C. Schepler

3000 B. C.   The Pyramids (Egypt). $3\frac{1}{7}$ (3.142857.....). The sides and
heights of the pyramids of Cheops and of Sneferu at Gizeh are in the
ratio 11:7, which makes the ratio of half the perimeter to the height
$3\frac{1}{7}$. In 1853, H. C. Agnew, Esq., London, published a letter from
Alexandria on evidence of this ratio found in the pyramids being
connected or related to the problem of the quadrature of the circle.

There is considerable disagreement in the dates given for the
construction of the pyramids, particularly that of the Great Pyramid[1].
Dates for pyramids as taken from various sources range from 1800 B. C.
to 4750 B. C. [7], 328; [11], 43; Encyclopedia Britannica; Americana
Encyclopedia; Chronology of World Events.

2000 B. C.   Rhind Papyrus (Egypt). $(\frac{16}{9})^2$ (3.1604.....). Also called
the Ahmes Papyrus after Ahmes, its writer. The rule given for con-
structing a square having the same area as a given circle is: *Cut
$\frac{1}{9}$ off the circle's diameter and construct a square on the remainder.*

The Rhind Papyrus is the oldest mathematical document in existence
and was founded on an older work believed by Birch to date back as
far as 3400 B. C. A. Henry Rhind, an Egyptologist, brought the manu-
script to England in the middle of the 19th century. It was deciphered
by Eisenlohr in 1877 and is part of the Rhind collection of the British
Museum.

Dates from various sources for the origin of the Rhind Papyrus
range from c. 1500 B. C. to c. 2000 B. C. [5], 9; [11], 261; [18],
14; [20], 3, 188; [24], 392.

950 B. C. Bible (Hebrews). 3. I Kings vii 23: ''And he made a molten
sea, ten cubits from the one brim to the other; it was round all
about..... and a line of thirty cubits did compass it round about''.
See also, II Chron. iv 2. The first book of Kings which deals with
Samuel quite some time before Solomon, was probably written about
932-800 B. C.[2]

The Babylonians, Hindus, and Chinese used the value 3. It is
probable that the Hebrews adopted this value from the Semites (Babylon-
ian predecessors). No definite statement for the value of π has yet
been found on the Babylonian cylinders (1600 to 2300 B. C.).

The value 3 is still more plainly given in the Talmud, where we
read that ''that which measures three lengths in circumference is

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[1] The Pyramid of Cheops, called Khufu.

[2] From personal communications with Brothers Reginald and Edmund,
University of Notre Dame.

one length across.'' The Talmud was not put into final form until
about 200 A. D., but it was patterned after the Old Testament, and
hence the value 3 was retained.[2] [5], 88; [11], 58, 261; [15], 50;
[18], 13; [22], 105; [24], 391.

460 B. C.   Hippocrates of Chios (Greece) attempted to square the
circle. He actually squared the lune and was the first to square a
curvilinear figure in attempting to square the circle. That he really
committed the fallacy of applying his lune-quadratures to the quadrature
of the circle is not generally accepted. Euclid's *Elements* was probably
founded on the first elementary textbook on geometry, written by
Hippocrates. [5], 22; [8], 310; [19], 41, 44; [20], 258; [24], 393.

440 B. C. Anaxagoras (499-428 B. C.) (Athens, Greece), a contemporary
of Hippocrates, ''drew the quadrature of the circle'' while in prison.
No solution was offered, however. This is the first mention found of
the quadrature problem as such. Anaxagoras was the last and most
famous philospher of the Ionian School. He was accused of heresy
because of his hypothesis that the moon shone only by reflected light
and that it was made of some earthly substance, while the sun, on
the contrary was a red hot stone that emitted its own light; born
in Asia Minor near Smyrna, he spent the greater part of his life in
Athens. [5], 17; [8], 299; [20], 256, 258; [22], 105.

430 B. C.   Antiphon (born 480 B. C.) of Rhamnus (Attica, Greece)
thought he had squared the circle. By inscribing polygons of ever
increasing numbers of sides, he pioneered the invention of the modern
calculus by approximately exhausting the difference between the polygon
and the circle, thus approximating the area of the circle. Bryson of
Heraclea, a contemporary of Antiphon, advanced the problem of the
quadrature considerably by circumscribing polygons at the same time
that he inscribed them. He erred, however, in assuming that the area
of a circle was the arithmetical mean between circumscribed and in-
scribed polygons. [5], 23; [19], 41; [20], 259; [24] 393.

425 B. C.   Hippias (born 460 B. C.) of Elis, (Greece), mathematician,
astronomer, natural scientist, and a contemporary of Socrates; invented
the quadratrix which he used to trisect the angle. Dinostratus used
this curve in 350 B. C. to square the circle. See 350 B. C., Dino-
stratus, [5], 20, 42, 50; [8], 310; [20], 259; [24], 392.

414 B. C.   Aristophanes (c.448-c.385 B. C.) (Athens, Greece), the
comic poet, in his play, *The Birds*, refers to a geometer who announces
his intention to make a square circle. Since that time the term
''circle-squarers'' has been applied to those who attempt the imposs-
ible. The Greeks had a special word τετραγωνιζειν, which meant ''to
busy oneself with the quadrature.'' [5], 17; [18], 12; [20], 257;
[22], 99.

390 B. C. (?)   Plato (429-348 B. C.) (Athens, Greece),called solutions

of the famous geometrical problems mechanical and not geometrical when they required the use of instruments other than ruler and compasses. He solved the duplication mechanically; founded the Academy and contributed to the philosophy of mathematics. [5], 27; [8], 316.

370 B. C.   Eudoxus (c.408 B. C.-c.355 B. C.) of Cnidus, (Greece) carried Antiphon's method of exhaustion further by considering both the inscribed and circumscribed polygons. He was a pupil of Archytas and Plato. In his astronomical observations he discovered that the solar year is longer than 365 days by six hours. Vitruvius credits him with inventing the sun-dial. See 430 B. C., Antiphon. [8], 306; [20]. 259.

350 B. C.   Dinostratus (Greece) used the quadratrix invented by Hippias to square the circle. See 425 B. C., Hippias. Menaechmus (c. 350 B. C.), brother of Dinostratus and a friend of Plato, invented the conic sections with which he solved the duplication. [5], 20, 27, 282; [8], 305; [24], 392.

300 B. C.   Euclid (Greece) offered no solution to the quadrature. He was the most successful textbook writer the world has ever known. Over 1000 editions of his geometry have appeared in print since 1482 and manuscripts of this work had dominated the teaching of the subject for 1800 years preceding that time. He was a pupil of Plato and studied in Athens; was a Greek mathematician and founder of the Alexandrian School. See c. 470 B. C., Hippocrates. [1], 340; [18], 21; [20], 268.

240 B. C.   Archimedes (278-212 B. C.) (Greece) gave between $3\frac{10}{71}$ (3.1408.....) and $3\frac{1}{7}$ (3.1428.....). His *Measurement of the Circle* contains but three propositions. (1) He proves that the area of a circle = πr^2; (2) shows that $\pi r^2:(2r)^2::11:14$, very nearly; (3) shows that the true value of π lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. For the latter, inscribed and circumscribed polygons of 96 sides each were used. He also gave the quadrature of the parabola; was an engineer, architect, physicist, and one of the world's greatest mathematicians.[3] [8], 299; [11], 261; [18], 14; [19], 100; [20], 11, 188.

213 B. C.   Chang T'sang (China). 3. Given in the *Chiu-chang Suan-shu* *(Arithmetic in Nine Sections)*, commonly called the *Chiu-chang*, the most celebrated Chinese text on arithmetic. Neither its authorship nor the time of its composition is known definitely. By an edict of the despotic emperor Shih Hoang-ti of the Ch'in Dynasty ''all books were burned and all scholars were buried in the year 213 B. C.'' After the death of this emperor, learning revived again. Chang T'sang, a scholar,

[3]The achievements of Archimedes were indeed remarkable since neither our present numerals nor the Arabic numerals, nor any system equivalent to our decimal system was known in his time. Try multiplying XCVII and MDLVIII in the Roman numeral system without using Arabic or common numerals. This will indicate the difficulties under which Archimedes labored.

found some old writings upon which he based the famous treatise, the *Chiu-chang*. [5], 71.

100 B. C.   Heron (Hero) of Alexandria, sometimes used the value $3\frac{1}{7}$ (3.1428.....) for purposes of practical measurement, and sometimes even the rougher approximation 3. He contributed to mensuration and was called the father of surveying. This date is also given as c. 50 A. D. Heath, however, believes that Hero may have lived considerably later, perhaps even in the fourth century A. D. [1], 340; [8], 309; [20], 232; [22], 108.

20 B. C.   Vitruvius (Marcus Vitruvius Pollio) (Rome, Italy) $3\frac{1}{8}$ (3.125). Roman architect and engineer; first described the direct measurement of distances by the revolution of a wheel. [8], 321; [20], 188, 238.

25 A. D.   Liu Hsiao (China). 3.16. He was of the Imperial House of the Han Dynasty and one of the most prominent ''circle squarers'' of his day. His son, Liu Hsing, devised a new calendar. [11], 261.

97 A. D.   Frontius (Sextus Julius Frontinus) (c. 40-103 A. D.) (Rome, Italy) used $3\frac{1}{7}$ (3.1428.....) for water pipes and areas. He said: ''The square digit is greater than the circle digit by $\frac{3}{14}$th of itself; the circle digit is less than the square digit by $\frac{3}{11}$''. A Roman author, soldier, surveyor, and engineer; was successively city preator of Rome, governor of Britain, and superintendent of the aqueducts at Rome. [8], 306; [15], 50.

125 A. D.   Ch'ang Höng (78-139 A. D.) (China). $\sqrt{10}$ (3.162.....). This was one of the earliest uses of this approximation. Höng was chief astrologer and minister under the emperor An-ti'; constructed an armillary sphere and wrote on astronomy and geometry. [23], 553; [24], 394.

150 A. D.   Ptolemy (Claudius Ptolemaeus) (87-165 A. D.) (Alexandria) $3\frac{17}{120}$ (3.141666.....). This value is found in his astronomical treatise, the *Syntaxis* (*Almagest* in Arabian[4]) in 13 books. The result was expressed in sexagesimal system as $3°8'30''$, *i.e.*, $3 + \frac{8}{60} + \frac{30}{60^2}$, which reduces to $3\frac{17}{120}$. This is the most notable result after Archimedes. Ptolemy was a geographer, mathematician, and one of the greatest of Greek astronomers. [1], 340; [5], 46; [8], 317; [18], 15; [20], 188; [24], 394.

250 A. D.   Wang Fan (Wang Fun) (229-267 A. D.) (China) $\frac{142}{45}$ (3.155...). Chinese astronomer. [11], 261; [23], 553; [24], 394.

263.   Liu Hui (China). $\frac{157}{50}$ (3.1400). Calculated the perimeters of

[4]*Almagest* is the usual nomenclature for the *Syntaxis*, derived from an Arabic term signifying ''the greatest''.

regular inscribed polygons up to 192 sides in the same way as Antiphon (430 B. C.); made a commentary on Chang T'sang's Chiu-chang (See 213 B. C., Chang T'sang); contemporary writer of Wang Fan and best-known Chinese mathematician of the 3rd century. [5], 71; [24], 394.

450.   Wo (China) 3.1432+. Geometer. [23], 553.

480.   Tsu Ch'ung-chih (430-501 A. D.) (China). Between 3.1415926 and 3.1415927. Expert in mechanics and interested in machinery; gave $\frac{22}{7}$ (3.1428.....) as an "inaccurate" value and $\frac{355}{113}$ (3.1415929.....) as the "accurate" value. The latter is often attributed to Adriaen Anthonisz. See 1585, Adriaen Anthonisz, and 1573, Valentin Otto. [11], 261; [20], 73; [23], 553; [24], 394.

500.   Arya-Bhata (Aryabhata) (c. 475-c. 550) (India) gave $3\frac{177}{1250}$ (3.1416) and $\frac{62832}{20000}$ (3.1416). The latter was calculated from the perimeter of an inscribed polygon of 384 sides[5]. Arya-Bhata was a Hindu mathematician who wrote chiefly on algebra, including quadratic equations, permutations, indeterminate equations, and magic squares.[1],341; [5],85,87;[6],10;[8],299;[11],261;[20],16;[23],553;[24],394.

505..   Varāhamihira Pancha Siddhāntikā (India) $\sqrt{10}$ (3.162.....). Hindu astronomer; the most celebrated of astronomical writers in early India; taught the sphericity of the earth and was followed in this respect by most other Hindu astronomers of the middle ages. [5], 96.

510.   Anicius Boethius (c. 480-524 A. D.) (Rome) said the circle had been squared in the period since Aristotle's time but noted that the proof was too long for him to give. A philosopher, statesman, and founder of medieval scholasticism; wrote on arithmetic and translated and revised many Greek writings on mathematics, mechanics, and physics. His *Consolations of Philosophy* were composed while in prison. He was executed at Pavia in 524. [8], 301; [20], 261.

530.   Baudhayana (India) $\frac{49}{16}$ (3.062.....). He lived before 530 A. D. His value and other of his works were published in England in 1875. [1], 341.

628.   Brahmagupta (Born 598 A. D.) (India). $\frac{22}{7}$ (3.1428.....). He gave 3 as the "practical value", and $\sqrt{10}$ (3.162.....) as the "exact value", perhaps because of the common approximate formula $\sqrt{a^2 + r} = a + \frac{1}{2a + r}$ which leads to $\sqrt{10} = 3 + \frac{1}{7}$, or the common Archimedean value. The value $\sqrt{10}$ was extensively used in medieval times. Brahmagupta, the most prominent Hindu mathematician of the 7th century, was the first Indian writer to apply algebra extensively to astronomy. [5]; 85; [8], 302; [20], 188; [24] 394.

This article will be concluded in the March-April issue.

[5]This may be due to a later writer by the same name.

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